At most two smooth holomorphic fibrations over hyperbolic curves

Abstract

We prove that a connected compact complex surface admits at most two holomorphic submersions with connected fibres onto compact curves of genus at least two, up to isomorphism of the bases. In particular, there is no surface with three distinct Kodaira fibrations, answering the existence question posed by Catanese (Jpn. J. Math., 2017). The result also includes isotrivial fibrations and requires no projectivity hypothesis. The proof relates the ramification of a pair of fibrations to an obstruction to a third. Equality of pulled-back hyperbolic metrics forces each reduced ramification component to be smooth and unramified over both bases. Its neighbourhood carries a cyclic action, and its normal bundle has negative degree. A third fibration would produce a holomorphic splitting whose cyclic average is tangent to the ramification curve. When all three fibrations are tangent there, the splittings are meromorphic; averaging and a constant linear combination cancel their principal parts. In either case the resulting regular foliation has the ramification curve as a compact leaf, contradicting the degree-zero condition imposed by its normal connection. The unramified case follows from factorization through a finite product cover.

Note. This paper was generated entirely by AI, including MiMo, using an automated research pipeline developed by Chenghua Liu and Hanyu Li.

1 Introduction

A Kodaira fibration is a smooth, non-isotrivial holomorphic family of curves over a compact curve: its fibres vary in complex structure, although the underlying map is a differentiable fibre bundle. Kodaira's construction [6] and the double Kodaira fibrations studied by Catanese and Rollenske [4] show that a single compact complex surface can support two such families. Catanese [3, Question 10] asked whether three distinct Kodaira fibrations can occur on the same surface. Salter [8, Question 3.3] asked a related question about smooth surface bundle structures on complex surfaces, and more generally on four-manifolds with nonzero signature.

There are restrictions on a possible third fibration and exact counts for particular surfaces. Chen [5, Theorems 1.1–1.2] proved that the Atiyah–Kodaira manifold considered there, and regular finite covers of products of hyperbolic surfaces, have exactly two surface bundle structures up to π_1-fibrewise diffeomorphism. Llosa Isenrich and Py [7, Theorem 2] proved that three distinct Kodaira fibrations on a compact complex surface would induce a finite-index image in the product of the three base fundamental groups. That theorem restricts such a configuration but does not exclude it. We exclude it by studying the ramification curves of the pairwise maps to products.

Throughout, a complex surface is a connected complex manifold of complex dimension two, and a curve is a connected compact Riemann surface. A curve is *hyperbolic* if its genus is at least two. A *fibration* is a surjective holomorphic map with connected fibres; it is *smooth* if it is a submersion. Two fibrations $f_i : X \to B_i$ are *equivalent* if $f_2 = \alpha \circ f_1$ for an isomorphism $\alpha : B_1 \to B_2$. No automorphism of X is included in this equivalence relation.

Theorem 1.1. *A compact complex surface admits at most two equivalence classes of smooth fibrations onto hyperbolic curves.*

The bound is attained by the projections of a product of two hyperbolic curves, and also by double Kodaira fibrations [4].

Corollary 1.2. *A compact complex surface admits at most two inequivalent Kodaira fibrations.*

Proof. The base of a Kodaira fibration has genus at least two; see [7, Introduction]. Theorem 1.1 therefore applies. $\square$

The proof has two geometric steps. First, the joint map of two inequivalent smooth fibrations is finite. On the normalization of any ramification component, the two restricted maps have the same ramification divisor. Uniqueness of their pulled-back hyperbolic metrics then identifies the local ramification image with the graph of a hyperbolic isometry. We deduce that the reduced ramification curves are smooth and pairwise disjoint. A component D with ramification index n has a cyclic neighbourhood model and satisfies

$$\mathcal{N}_{D/X}^{\otimes n} \cong TD, \qquad \deg \mathcal{N}_{D/X} = -\frac{2g(D)-2}{n} < 0.$$

These statements concern ramification curves in the source. They do not assert that the entire branch divisor in the product is smooth: different points of the source may lie over the same branch value.

Second, a third fibration would force a regular holomorphic foliation tangent to D. Schwarz–Pick rigidity shows that its tangency with the first two fibrations occurs either everywhere on D or nowhere on D. In the latter case, cyclic averaging of a holomorphic splitting produces the required foliation. In the former, two meromorphic splittings have principal coefficients whose ratio is constant on D. Averaging removes the intermediate Laurent terms, and an affine combination cancels the remaining pole. A compact leaf of a regular holomorphic foliation has a degree-zero normal bundle, giving the contradiction. This degree obstruction is classical in the theory of holomorphic connections [2, 1]; the construction of the splitting from the three fibrations is the step needed here. Appendix A gives an alternative calculation of the common-tangency obstruction. If the original pair is unramified, an elementary factorization argument on a finite product cover completes the proof.

The theorem counts holomorphic submersions for the given complex structure on X. It is not a bound on arbitrary smooth surface bundle structures on four-manifolds, whose number can be arbitrarily large [8, 9]. Hyperbolicity of the bases and smoothness of each fibration are essential to the argument. In particular, smoothness must be checked separately for the two projections of a branched cover; Remark 2.5 explains this distinction for a construction appearing in the literature.

2 Ramification and hyperbolic metrics

For a smooth compact curve D in a complex surface X, write $\mathcal{N}_{D/X} = TX|_D/TD$ for its normal bundle, so that $D^2 = \deg \mathcal{N}_{D/X}$. We first establish the finiteness of a pair map and recall the degree obstruction used later.

Lemma 2.1. *If $f : X \to B$ and $g : X \to C$ are inequivalent smooth fibrations from a compact complex surface, then $(f, g) : X \to B \times C$ is finite and surjective.*

Proof. The fibres of f are homologous, so the topological degree of the restriction of g to a fibre of f is independent of the fibre. A holomorphic map of compact curves has degree zero exactly when it is constant. Thus, if g is constant on one fibre of f, it is constant on every fibre. Local holomorphic sections of f then give a holomorphic factorization $g = \alpha \circ f$. The map $\alpha : B \to C$ is nonconstant.

Over a regular value of α, a fibre of g is the disjoint union of $\deg \alpha$ fibres of f. Its connectedness forces $\deg \alpha = 1$, contrary to inequivalence.

Consequently g is nonconstant on every fibre of f. Each such fibre is a smooth connected, hence irreducible, compact curve, so no curve can be contracted by (f, g). Its fibres are therefore zero-dimensional compact analytic sets, hence finite. The map is proper, so it is finite. Its image is a closed analytic subset of dimension two in the connected surface $B \times C$, and is therefore the whole product. □

A holomorphic connection on a line bundle L over a curve is a $\mathbb{C}$-linear sheaf map $\nabla : L \to \Omega^1 \otimes L$ satisfying $\nabla(hs) = \mathrm{d}h \otimes s + h\nabla s$ for local holomorphic functions h and sections s. The following form of the connection obstruction will suffice; compare [2] and [1, Theorem 7.1 and Corollary 7.3].

Lemma 2.2. *A holomorphic line bundle on a compact curve that admits a holomorphic connection has degree zero. In particular, if a regular holomorphic foliation on a neighbourhood of a smooth compact curve D has D as a leaf, then* $\deg \mathcal{N}_{D/X} = 0$*.*

Proof. Choose a nonzero meromorphic section s of the line bundle and write $\nabla s = \omega \otimes s$. If $s = z^m ue$, where e is a holomorphic frame and u is a holomorphic unit, then

$$\omega = m\frac{\mathrm{d}z}{z} + \frac{\mathrm{d}u}{u} + \theta, \qquad \nabla e = \theta \otimes e,$$

with θ holomorphic. Thus the poles of ω have residues equal to the orders of the zeros and poles of s. The residue theorem gives

$$\deg L = \sum_p \mathrm{ord}_p(s) = \sum_p \mathrm{Res}_p\, \omega = 0.$$

For the second assertion, choose foliation charts (z, w) with $D = \{w = 0\}$ and leaves given by $w =$ constant. On a connected overlap the transverse coordinates satisfy $w' = \psi(w)$, with $\psi(0) = 0$ and $\psi'(0) \neq 0$. The normal transition functions $\psi'(0)$ are constant. They define a flat holomorphic connection on $\mathcal{N}_{D/X}$, namely the normal Bott connection. The first assertion applies. □

For a hyperbolic curve B, let h_B denote its metric of curvature -1. Pullback by an unramified covering gives the curvature -1 metric on the covering curve. We also need uniqueness when two pullbacks have the same degeneracy divisor.

Lemma 2.3. *Let $u : D \to B$ and $v : D \to C$ be nonconstant maps of curves, with $g(B), g(C) \geq 2$. If their ramification divisors on D coincide, then $u^*h_B = v^*h_C$. Near every point of D, the image of (u, v) is contained in the graph of a holomorphic local hyperbolic isometry from B to C.*

Proof. In a local coordinate z on D, write the two metrics as $\rho_u|\mathrm{d}z|^2$ and $\rho_v|\mathrm{d}z|^2$. At a point with common ramification order m, both densities are $|z|^{2m}$ times a smooth positive function. Hence

$$w = \log(\rho_u/\rho_v)$$

is a smooth real function on all of D, independent of the coordinate. Away from the common zeros, the curvature equation gives

$$\Delta_0 w = 2(\rho_u - \rho_v) = 2\rho_v(e^w - 1),$$

where Δ_0 is the Euclidean Laplacian in z. Both sides extend smoothly across the zeros. If ω_u and ω_v are the pulled-back area forms, this equation is $i\partial\bar{\partial}w = \omega_u - \omega_v$. Multiplication by w and integration by parts yield

$$-\int_D i\partial w \wedge \bar{\partial} w = \int_D w(e^w - 1)\omega_v \geq 0.$$

The left-hand side is nonpositive, so w is constant. Since ρ_v is positive away from finitely many points, the curvature equation then gives $w = 0$.

Where u is locally invertible, $v \circ u^{-1}$ is a holomorphic local hyperbolic isometry. Lift small target charts to the unit disc. Such an isometry agrees locally with the disc automorphism having the same value and differential: isometries with the same first-order data agree along geodesics from that point. For a neighbourhood of a ramification point of D, choose a nearby unramified point and the corresponding disc automorphism A. The lifted maps satisfy $\widetilde{v} = A \circ \widetilde{u}$ on a nonempty open set, hence on the whole neighbourhood by the identity theorem. Descending to smaller target charts proves the graph assertion at the ramification point as well. □

For two inequivalent smooth fibrations $f : X \to B$ and $g : X \to C$, define

$$R(f, g) = \operatorname{div}(\mathrm{d}f \wedge \mathrm{d}g).$$

The wedge is a nonzero holomorphic section of $K_X \otimes f^* K_B^{-1} \otimes g^* K_C^{-1}$, where K denotes the canonical line bundle. Thus $R(f, g)$ is an effective divisor, possibly empty. Its support is the tangency locus of the two fibre foliations.

Proposition 2.4. *Let $f : X \to B$ and $g : X \to C$ be inequivalent smooth fibrations from a compact complex surface to hyperbolic curves. The reduced support of $R(f, g)$ is a disjoint union of smooth compact curves D, and $f|_D$ and $g|_D$ are unramified coverings. If D has coefficient $n-1$ in $R(f, g)$, then $n \geq 2$ and*

$$\mathcal{N}_{D/X}^{\otimes n} \cong TD, \qquad D^2 = -\frac{2g(D) - 2}{n} < 0. \tag{1}$$

There is a neighbourhood U of D on which f and g lift to holomorphic retractions $r, s : U \to D$. Moreover, U admits a cyclic action of order n fixing D pointwise and preserving r and s. Locally along D, suitable coordinates give

$$r = x, \qquad s = x + y^n, \qquad \tau(x, y) = (x, \zeta y), \qquad \zeta = e^{2\pi i/n}. \tag{2}$$

Here the same coordinate on D is used for both retractions.

Proof. Let $\nu : \widehat{D} \to X$ be the normalization map of an irreducible component of $R(f, g)$. The cotangent maps induced by f and g are nowhere-zero maps of line bundles into $\nu^*\Omega^1_X$. Their images coincide along $\widehat{D}$, so they differ by a nowhere-zero holomorphic bundle isomorphism. Composing with $\nu^*\Omega^1_X \to \Omega^1_{\widehat{D}}$ shows that $\mathrm{d}(f \circ \nu)$ and $\mathrm{d}(g \circ \nu)$ are related by the same isomorphism. If either restricted map were constant, both would be constant, contradicting Lemma 2.1. They are therefore nonconstant and have identical ramification divisors.

By Lemma 2.3, each local branch of the ramification image is a graph of a local hyperbolic isometry. Fix a point p of the ramification divisor. All source branches through p give the same graph germ: its value is $(f(p), g(p))$, and its derivative is the unique scalar relating the nonzero covectors $\mathrm{d}f_p$ and $\mathrm{d}g_p$ in local target coordinates. To obtain the derivative even when the source branch is singular, take a limit from points of its normalization where the restricted differentials are nonzero. A local hyperbolic isometry is determined by its value and differential, so the graph germs agree.

We now apply this conclusion to the finite germ of (f, g) at p. Choose disjoint source neighbourhoods around the finitely many points over $(f(p), g(p))$, and shrink a target bidisc so that its inverse image is contained in their union. The component representing the germ at p is finite over the bidisc, has only p over its centre, and may be chosen connected. After further shrinking, its branch locus is contained in the single graph just identified. This assertion concerns only this source representative; branch images from other points over the centre need not agree with it.

Straighten the graph to $\{v = 0\}$. Away from its inverse image, the representative is an unramified cover of $\Delta \times \Delta^*$, where Δ is a disc and $\Delta^* = \Delta \setminus \{0\}$. This cover is connected: a proper analytic divisor in a connected complex manifold does not disconnect it, since paths can be perturbed off its real-codimension-two strata. The fundamental group of $\Delta \times \Delta^*$ is $\mathbb{Z}$, so each connected finite cover is isomorphic to

$$(x, y) \longmapsto (x, y^n) \quad \text{over } \Delta \times \Delta^*.$$

The isomorphism is holomorphic because both covering maps are local biholomorphisms. It extends across the divisor by uniqueness of the normal finite extension: both extensions are obtained by integral closure of the target structure sheaf in the same covering field. Since the source is smooth, it is normal. Thus this is also the local form of the original finite map. Its reduced ramification curve is smooth and maps isomorphically to the local branch graph. In particular, different ramification components cannot meet, and both restrictions to each component are unramified. The local index n is the coefficient of that component in $R(f, g)$ plus one.

Fix a component D. Choose a tubular neighbourhood U which deformation retracts onto D. The induced image of $\pi_1(U)$ under f is the image of $\pi_1(D)$ under $f|_D$. The covering-space lifting criterion for the unramified covering $f|_D : D \to B$ therefore gives a lift $r : U \to D$ with $r|_D = \mathrm{id}_D$. This lift is holomorphic because the covering is locally biholomorphic. The same argument gives a holomorphic lift s of g, also restricting to id_D. The pair (r, s) is locally a cyclic cover branched over the diagonal; in local coordinates it consequently has the form $r = x$, $s = x + y^n$.

If $x' = \phi(x)$ is a change of coordinate on D, the corresponding source coordinates satisfy the exact identities

$$x' = \phi(x), \qquad (y')^n = \phi(x + y^n) - \phi(x). \tag{3}$$

Writing $y' = \lambda(x)y + O(y^{n+1})$ gives $\lambda(x)^n = \phi'(x)$. The normal coordinate changes therefore have nth powers equal to the tangent coordinate changes on D, proving $\mathcal{N}_{D/X}^{\otimes n} \cong TD$. Taking degrees gives (1); here $g(D) \geq 2$ because D covers a hyperbolic curve.

To glue the cyclic action, factor the second identity in (3) as

$$(y')^n = y^n H(x, y^n), \qquad H(x, 0) = \phi'(x) \neq 0.$$

On each sufficiently small connected overlap, a holomorphic nth root of H gives $y' = \xi y H(x, y^n)^{1/n}$ for a constant nth root of unity ξ. Every transition thus commutes with $y \mapsto \zeta y$. The coordinate actions glue as germs along D, preserve r, s, and have order n. For completeness, these germs can be realized on an invariant open neighbourhood. Compactness gives representatives of their finitely many powers on a common neighbourhood V. Choose a smaller neighbourhood W of D such that all their images lie in V and all group composition identities hold on W. The union of these images of W is invariant, because composing any two representatives on W gives another power. Restricting to this union yields the asserted action on an actual neighbourhood. □

Remark 2.5 (Checking both projections). A branched cover which gives one smooth fibration need not give a second. For a cyclic cover branched along a smooth graph $z = \phi(t)$, the local map is

$$(t, w) \longmapsto (t, w^m + \phi(t)), \qquad m \geq 2.$$

The first projection is a submersion. The differential of the second is $mw^{m-1}\mathrm{d}w + \phi'(t)\mathrm{d}t$, which vanishes at a ramification point with $\phi'(t) = 0$.

This distinction matters for the branch-cover construction discussed in [4, Remark 2.6], where the branch divisor is *étale* over one factor but ramified over the other. In the intermediate genus-six family of [10, pp. 186–187], a genus-nine curve F has degree-two maps $\pi_i : F \to C_3$ to a genus-three curve. After an unramified base change $B \to F$, the family is a double cover of $B \times C_3$ branched along the two resulting graphs. Hurwitz's formula gives

$$\deg \operatorname{div}(\mathrm{d}\pi_i) = (2 \cdot 9 - 2) - 2(2 \cdot 3 - 2) = 8.$$

This ramification persists under the unramified base change, and the local calculation above shows that the projection of this intermediate family to C_3 is not a submersion. The construction therefore does not supply two smooth projections. This observation concerns the intermediate family, not Zaal's subsequent construction of a family of genus-three curves.

3 Excluding a third fibration

Suppose that $f_i : X \to B_i$, $1 \le i \le 3$, are pairwise inequivalent smooth fibrations onto hyperbolic curves. Put $R_{ij} = R(f_i, f_j)$. We first show that a third fibration cannot be tangent to just part of a component of R_{12}.

Lemma 3.1. *Let D be a component of R_{12}. Either $D \cap (R_{13} \cup R_{23}) = \varnothing$, or D is a component of all three divisors. In the latter case its ramification index is the same for all three pairs.*

Proof. Suppose D meets a component E of R_{13} at p. By Proposition 2.4, $f_1|_E$ and $f_3|_E$ are unramified coverings, and hence pull back the base metrics to h_E. The covectors $\mathrm{d}f_1$ and $\mathrm{d}f_3$ are proportional at p, so the two rank-one Hermitian forms $(f_1^* h_{B_1})_p$ and $(f_3^* h_{B_3})_p$ are proportional. Their common nonzero restriction to T_pE makes the proportionality factor one.

Since $f_1|_D$ is an unramified covering, the restriction of this identity to T_pD says that $f_3|_D$ attains equality in Schwarz–Pick at p. In particular it is nonconstant. The equality case of Schwarz–Pick, applied to its lift between universal covering discs, implies that $f_3|_D$ is an unramified covering.

Set $N = \mathcal{N}_{D/X}$, $\kappa_D = 2g(D) - 2$, and let n be the index of D for the pair (f_1, f_2). The bundle map on $TX|_D$ given by

$$\beta = \mathrm{d}f_3 - \mathrm{d}(f_3|_D) \circ \mathrm{d}(f_1|_D)^{-1} \circ \mathrm{d}f_1$$

vanishes on TD, and therefore induces a holomorphic map $N \to (f_3|_D)^* TB_3$. The bundle of such maps has degree

$$\deg \operatorname{Hom}\bigl(N, (f_3|_D)^* TB_3\bigr) = -\kappa_D - \deg N = -\kappa_D \left(1 - \frac{1}{n}\right) < 0.$$

A line bundle of negative degree has no nonzero holomorphic section, so $\beta = 0$. Hence $D \subset R_{13}$, and proportionality with $\mathrm{d}f_2$ gives $D \subset R_{23}$ as well. The same reasoning, with indices 1 and 2 interchanged, applies if D initially meets R_{23}. Distinct components of each R_{ij} are disjoint by Proposition 2.4.

In the common-component case, (1) applied to each pair states that $n_{ij} \deg N = \deg TD$. As $\deg N \neq 0$, all three indices n_{ij} are equal. $\square$

We will average splittings of a retraction $r : U \to D$ over a cyclic action preserving r. A splitting is a bundle map $\sigma : r^*TD \to TU$ satisfying $\mathrm{d}r \circ \sigma = \mathrm{id}$; it may also be meromorphic. If τ generates an action of order n, define

$$A(\sigma)_p = \frac{1}{n}\sum_{j=0}^{n-1} \mathrm{d}(\tau^j)_{\tau^{-j}p} \circ \sigma_{\tau^{-j}p}. \tag{4}$$

All terms have domain $T_{r(p)}D$, since $r \circ \tau = r$. Each term is again a splitting, so $A(\sigma)$ is an invariant splitting. The average preserves holomorphicity and, for meromorphic splittings, the bound on pole order along D.

Lemma 3.2. *The disjoint alternative in Lemma 3.1 is impossible.*

Proof. Use the retraction r and cyclic action associated to (f_1, f_2) by Proposition 2.4. Since $D \cap R_{13} = \varnothing$, the line bundle $\ker \mathrm{d}f_3$ is transverse to $\ker \mathrm{d}r$ along D. After shrinking to an invariant neighbourhood, the restriction of $\mathrm{d}r$ to $\ker \mathrm{d}f_3$ is an isomorphism. Its inverse gives a holomorphic splitting $\sigma : r^*TD \to TU$.

At any $p \in D$, the differential of τ has eigenvalue 1 on T_pD and eigenvalue $\zeta \neq 1$ on the vertical line $\ker \mathrm{d}r_p$. The invariant splitting $A(\sigma)$ therefore has image T_pD at p. Its image on U is a holomorphic line subbundle of TU. A line subbundle is integrable: if V is a local nonvanishing generator, then $[aV, bV] = (aV(b) - bV(a))V$. It thus defines a regular holomorphic foliation with compact leaf D. Lemma 2.2 gives $\deg \mathcal{N}_{D/X} = 0$, contradicting (1). □

In the common-component case, the fibre directions instead give meromorphic splittings. The cyclic action leaves only one possible negative Laurent power, which can be cancelled by an affine combination.

Lemma 3.3. *The common-component alternative in Lemma 3.1 is impossible.*

Proof. Lift the three maps on a tubular neighbourhood of D to retractions $r_i : U \to D$ with $r_i|_D = \mathrm{id}_D$, as in the proof of Proposition 2.4. Shrink U so that no other component of any R_{ij} meets it. Let $n \geq 2$ be the common ramification index. Use the cyclic action for (r_1, r_2), and local coordinates with

$$r_1 = x, \qquad r_2 = x + y^n.$$

The derivative $\partial r_3/\partial y$ has order exactly $n-1$ along D, and $r_3(x, 0) = x$. Consequently

$$r_3 = x + a(x)y^n + O(y^{n+1}), \qquad a(x) \neq 0. \tag{5}$$

On changing coordinates as in (3), comparison of the coefficients of y^n in $r_3' = \phi(r_3)$ gives $\widetilde{a}(\phi(x))\phi'(x) = \phi'(x)a(x)$, where $\widetilde{a}$ denotes the coefficient in the new chart. Thus the coefficients define a holomorphic function on the compact curve D, and this function is a constant $a \neq 0$. Moreover,

$$\mathrm{d}r_2 \wedge \mathrm{d}r_3 = \big(n(a-1)y^{n-1} + O(y^n)\big)\mathrm{d}x \wedge \mathrm{d}y.$$

The common index for (r_2, r_3) is n, so $a \neq 1$.

Away from D, the line bundles $\ker \mathrm{d}r_2$ and $\ker \mathrm{d}r_3$ define splittings σ_2, σ_3 of $\mathrm{d}r_1$. They extend meromorphically across D, with poles of order at most $n-1$. In the chosen coordinates,

$$\begin{aligned}
\sigma_2(\partial_x) &= \partial_x - \frac{1}{n}y^{1-n}\partial_y, \\
\sigma_3(\partial_x) &= \partial_x - \frac{\partial r_3/\partial x}{\partial r_3/\partial y}\partial_y = \partial_x - \frac{1}{an}y^{1-n}\partial_y + O(y^{2-n})\partial_y.
\end{aligned}$$

The cyclic action preserves r_1 and r_2, so it fixes σ_2. Average σ_3 by (4).

Under the pushforward by $\tau(x,y)=(x,\zeta y)$, a vertical monomial $y^k\partial_y$ is multiplied by ζ^{1-k}. An invariant vertical Laurent series can therefore contain this monomial only when $k\equiv 1 \pmod n$. Among the integers from $1-n$ through 0, the only such integer is $1-n$. The leading monomial in σ_3 survives averaging, and hence

$$A(\sigma_3)(\partial_x)=\partial_x-\frac{1}{an}y^{1-n}\partial_y+O(y)\partial_y.$$

Since a is a global constant different from 1, the bundle map

$$\sigma=\frac{aA(\sigma_3)-\sigma_2}{a-1} \tag{6}$$

is globally defined. Its pole cancels in every chart, so it is holomorphic across D. The coefficients in this affine combination sum to one, giving $\mathrm{d}r_1\circ\sigma=\mathrm{id}$. It is invariant and has image TD along D; equivalently, its local expression is $\sigma(\partial_x)=\partial_x+O(y)\partial_y$.

The image of σ defines a regular holomorphic foliation with compact leaf D. Lemma 2.2 contradicts $\deg\mathcal{N}_{D/X}<0$. $\square$

It remains to treat an unramified pair. The following holomorphic factorization argument is sufficient here; Chen [5, Theorem 1.2] treats surface bundle structures on regular finite covers of products in the stronger topological setting.

Lemma 3.4. *Let $q:X\to B_1\times B_2$ be a finite unramified covering, with $g(B_1),g(B_2)\geq 2$. Suppose the induced factor maps $f_i=\mathrm{pr}_i\circ q$ have connected fibres. Every fibration $h:X\to C$ onto a hyperbolic curve is then equivalent to f_1 or f_2.*

Proof. Write $\Gamma_i=\pi_1(B_i)$ and identify $\pi_1(X)$ with a finite-index subgroup $\Gamma\leq\Gamma_1\times\Gamma_2$. Set

$$H_1=\{\gamma\in\Gamma_1:(\gamma,1)\in\Gamma\},\qquad H_2=\{\gamma\in\Gamma_2:(1,\gamma)\in\Gamma\}.$$

Each H_i has finite index in Γ_i, and $H_1\times H_2\subset\Gamma$. Let $B_i'\to B_i$ be the corresponding unramified curve coverings. Covering-space theory gives a finite unramified cover $p:B_1'\times B_2'\to X$ over $B_1\times B_2$. It is holomorphic because it is locally a lift through a biholomorphic covering chart.

Consider $F=h\circ p:B_1'\times B_2'\to C$. The restrictions $F_x:B_2'\to C$ have a common topological degree. If this degree is zero, every F_x is constant and F depends only on $x\in B_1'$. If the degree is positive, differentiation in a vector of T_xB_1' gives a holomorphic section of F_x^*TC on B_2'. Its degree is $\deg(F_x)(2-2g(C))<0$, so that section vanishes. This holds for every x and every such vector, hence F is independent of the first factor. In either case, F factors through one projection.

Since p is a local biholomorphism, this factorization implies that $\mathrm{d}h$ vanishes on $\ker\mathrm{d}f_i$ for one i. The map h is therefore constant on every connected fibre of f_i. Local holomorphic sections of f_i give $h=\alpha\circ f_i$ for a holomorphic map $\alpha:B_i\to C$. The map α is nonconstant, and connectedness of the fibres of h forces $\deg\alpha=1$, as in Lemma 2.1. Thus h and f_i are equivalent. $\square$

Proof of Theorem 1.1. Suppose three pairwise inequivalent smooth fibrations exist. If $R_{12}\neq 0$, choose one of its components D. Lemma 3.1 gives two alternatives, excluded by Lemmas 3.2 and 3.3. Hence $R_{12}=0$. By Lemma 2.1, (f_1,f_2) is then a finite unramified cover of $B_1\times B_2$. Lemma 3.4 makes f_3 equivalent to one of f_1,f_2, a contradiction. $\square$

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A The common-tangency obstruction in coordinates

We give a second proof of Lemma 3.3, starting after the constant $a \in \mathbb{C} \setminus \{0, 1\}$ has been obtained. It produces a holomorphic connection directly on TD, rather than constructing a foliation and using its normal connection.

In a chart write

$$r_1 = x, \qquad r_2 = x + y^n, \qquad r_3 = x + ay^n + \sum_{m>n} b_m(x) y^m.$$

For an overlap with coordinate change $x' = \phi(x)$, equation (3) gives

$$(y')^n = \phi'(x) y^n + \frac{1}{2}\phi''(x) y^{2n} + O(y^{3n}).$$

Let $\widetilde{b}_m(x')$ be the coefficients of r_3' in the new chart. Compare the coefficients of y^{2n} in

$$x' + a(y')^n + \sum_{m>n} \widetilde{b}_m(x')(y')^m = \phi\left(x + ay^n + \sum_{m>n} b_m(x) y^m\right).$$

For $n < m < 2n$, the first correction to the leading term of $(y')^m$ has order $m + n > 2n$, so these terms contribute nothing at order $2n$. Terms with $m > 2n$ also contribute nothing. Thus

$$\frac{a}{2}\phi''(x) + \widetilde{b}_{2n}(\phi(x))\phi'(x)^2 = \phi'(x) b_{2n}(x) + \frac{a^2}{2}\phi''(x),$$

or equivalently

$$\widetilde{b}_{2n}(\phi(x))\phi'(x)^2 = \phi'(x) b_{2n}(x) + \frac{a^2 - a}{2}\phi''(x). \tag{7}$$

The root-of-unity factor in the change of normal coordinate has no effect on this identity, since it is raised to the power $2n$.

Define a holomorphic function in each chart by

$$\Gamma_x = -\frac{2b_{2n}(x)}{a^2 - a}.$$

Equation (7) becomes

$$\Gamma_{x'}(\phi(x))\phi'(x)^2 = \phi'(x)\Gamma_x(x) - \phi''(x).$$

This is the compatibility condition for the local connections

$$\nabla_{\partial_x} \partial_x = \Gamma_x \partial_x.$$

Indeed, $\partial_{x'} = (\phi')^{-1}\partial_x$, and the Leibniz rule gives precisely the displayed transformation law. The local connections therefore glue to a holomorphic connection on TD. Lemma 2.2 would imply $\deg TD = 0$, whereas $\deg TD = 2 - 2g(D) < 0$. This proves the required contradiction.