Galois–modular extensions for Tannakian centers with prescribed spherical structure

Abstract

We prove that every premodular fusion category over $\mathbb{C}$ whose Müger center is Tannakian admits a fully faithful braided spherical embedding into a modular fusion category with integral centralizer. The spherical structure is prescribed and need not be positive on the center. Consequently, the embedded simple objects form a union of Galois orbits. This removes the pseudounitarity assumption from the Tannakian-center case of the extension theorem of Johnson–Freyd (arXiv, 2026). The spherical extension step concerns a transparent order-two object with trivial self-braiding and categorical dimension -1: every minimal nondegenerate extension admits a spherical structure extending the given one. We combine fixed-boundary extension theory with the twist constraint of Lacabanne (IMRN, 2021) for nondegenerate pivotal categories, which excludes the only possible nontrivial pivotal slope. Applying this result to a sphericalization yields a modular ambient category with weakly integral centralizer. We then use the sign character comparing its inherited and positive spherical structures to obtain integrality, either directly through a grading or after diagonal condensation. A pointed example shows why the prescribed spherical structure need not extend to a fixed ambient category.

Note. This paper was generated entirely by AI, including MiMo, using an automated research pipeline developed by Chenghua Liu and Hanyu Li.

1 Introduction

A modular extension of a premodular fusion category retains its braiding and spherical traces while placing it inside a category with nondegenerate modular data. The centralizer of the embedded category measures the complement introduced by the extension. Minimal extensions require this centralizer to be no larger than the original Müger center. The Galois–modular extension problem instead asks for an integral centralizer, allowing a larger ambient category in exchange for control of its arithmetic.

The connection with Galois theory is precise. Plavnik, Schopieray, Yu, and Zhang [8] prove that a fusion subcategory of a modular category is closed under the Galois action on simple objects if and only if its centralizer is integral. Thus an embedding with integral centralizer realizes the given simple objects as a union of ambient Galois orbits. Schopieray's conjecture, as formulated in [5], asks whether every premodular fusion category admits such an extension.

Johnson–Freyd [5] proves the conjecture for pseudounitary braided fusion categories equipped with their positive spherical structures. For a Tannakian center, his Proposition 2.3 already supplies a nondegenerate braided extension with integral centralizer without assuming pseudounitarity. The remaining issue is spherical: that extension need not come with a spherical structure restricting to the one specified on the original category. Even when a specified structure extends pivotally, the extension need not be spherical. Our result resolves this issue for Tannakian centers.

Throughout, fusion categories and tensor functors are over $\mathbb{C}$. A *premodular* category is a braided fusion category equipped with a specified spherical structure; it is *modular* if its Müger center is trivial. We write $\mathcal{Z}_2(\mathcal{C})$ for the Müger center and $\mathcal{D}'_{\mathcal{C}}$ for the centralizer of a fusion subcategory $\mathcal{D} \subseteq \mathcal{C}$.

A fusion category is *integral* if all its simple objects have integer Frobenius–Perron dimensions, and *weakly integral* if its total Frobenius–Perron dimension is an integer.

Theorem 1.1. *Let $\mathcal{B}$ be a premodular fusion category with a specified spherical structure, and suppose that*

$$\mathcal{Z}_2(\mathcal{B}) \simeq \mathrm{Rep}(G)$$

as braided fusion categories, for a finite group G. There exist a modular fusion category $\mathcal{M}$ and a fully faithful braided spherical tensor functor

$$\mathcal{B} \hookrightarrow \mathcal{M}$$

such that $\mathcal{B}'_{\mathcal{M}}$ is integral.

The equivalence with $\mathrm{Rep}(G)$ concerns the braiding, not the spherical structure: no positivity is assumed on the center. The resulting extension is not required to be minimal. By the criterion of [8], the image of $\mathrm{Irr}(\mathcal{B})$ is a union of Galois orbits in $\mathrm{Irr}(\mathcal{M})$.

The key extension result is Proposition 2.2. If the entire center is generated by an order-two object f with trivial self-braiding and dimension -1, then every minimal nondegenerate extension supports the specified spherical structure. There are two separate ingredients. Fixed-boundary extension theory extends the specified pivotal structure, including its identification on the subcategory. Lacabanne's twist constraint [6, Proposition 2.21] excludes the nontrivial central object as a possible pivotal slope, so this extension is spherical.

Sphericalization reduces the general construction to this order-two case, but enlarges the integral centralizer to a weakly integral one. The latter has both an inherited spherical structure and a canonical positive spherical structure. Their ratio is a sign character. When this character is trivial on the common Tannakian center, it produces an invertible object in the odd component of a grading, forcing integrality. Otherwise, its positive kernel can be matched with a pseudounitary Galois–modular extension. Diagonal condensation then produces an integral centralizer while leaving the prescribed structure on $\mathcal{B}$ unchanged. Appendix B illustrates why changing the ambient extension can be necessary.

We use the centralizer and double-centralizer results of [7, 2], and the fusion-category conventions of [3, 2]. The symbols $\mathrm{Irr}(\mathcal{C})$ and $\mathrm{Inv}(\mathcal{C})$ denote, respectively, the isomorphism classes of simple and invertible objects. We identify a fusion category with the image of a fully faithful tensor functor when no confusion can arise. For a spherical braided category, $\mathcal{C}^{\mathrm{rev}}$ has the reversed braiding, the same spherical structure, and inverse twist.

2 Spherical extension over $\mathrm{Rep}(\mathbb{Z}/2)$

The obstruction to sphericality of a pivotal structure is its slope. In a nondegenerate braided fusion category, the slope is represented by monodromy with an invertible object. The following reformulation of Lacabanne's result constrains the twist of that object.

Lemma 2.1 (Lacabanne)**.** *Let $\mathcal{C}$ be a nondegenerate braided fusion category with a pivotal structure j and associated balancing θ. For $X \in \mathrm{Irr}(\mathcal{C})$, put*

$$d_X = \dim_R^j(X), \qquad n_X = d_X d_{X^*}, \qquad \gamma_X = \frac{d_X}{d_{X^*}}.$$

The scalars γ_X define a tensor automorphism of $\mathrm{id}_{\mathcal{C}}$. There is a unique $t \in \mathrm{Inv}(\mathcal{C})$ such that

$$c_{X,t}c_{t,X} = \gamma_X \, \mathrm{id}_{t\otimes X} \qquad \textit{for every simple } X, \tag{2.1}$$

and

$$\theta_{t^{-1}} = 1. \tag{2.2}$$

Proof. Let $\delta : \mathrm{id} \to (-)^{****}$ be the canonical Radford isomorphism. The pivotal slope is the tensor automorphism $\delta^{-1}j^{**}j$, whose scalar on a simple object is γ_X. Nondegeneracy gives the monodromy isomorphism

$$\mathrm{Inv}(\mathcal{C}) \simeq \mathrm{Aut}_{\otimes}(\mathrm{id}_{\mathcal{C}}). \tag{2.3}$$

Indeed, under the factorization equivalence $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}} \simeq \mathcal{Z}(\mathcal{C})$, the tensor unit with a half-braiding pulls back to a pair (t, t^{-1}) of invertible objects. The half-braiding corresponds to monodromy with t. This proves the existence and uniqueness in (2.1). The argument for (2.3) does not use a pivotal structure.

For completeness, we give a Gauss-sum proof of (2.2). The pivotal Hopf-link matrix

$$S_{XY} = \mathrm{tr}_R^j(c_{Y,X}c_{X,Y})$$

is invertible, and

$$\theta_{X^*} = \theta_X \gamma_X^{-1}; \tag{2.4}$$

these statements do not require sphericality [6, Propositions 2.10 and 2.17]. Set

$$\tau = \sum_{X \in \mathrm{Irr}(\mathcal{C})} \theta_X n_X.$$

Writing $N_{XY}^Z = \dim \mathrm{Hom}(Z, X \otimes Y)$, the balancing formula and Frobenius reciprocity give

$$\begin{aligned}\sum_X \theta_X d_{X^*} S_{XY} &= \theta_Y^{-1} \sum_Z \theta_Z d_Z \sum_X N_{XY}^Z d_{X^*} \\ &= \theta_Y^{-1} \sum_Z \theta_Z d_Z d_{Z^*} d_Y = \tau \frac{d_Y}{\theta_Y}.\end{aligned}$$

Here the second equality uses $N_{XY}^Z = N_{Z^*Y}^{X^*}$ and multiplicativity of pivotal dimensions. The vector $(\theta_X d_{X^*})_X$ is nonzero and S is invertible, so $\tau \neq 0$.

Reindexing the sum for τ by duality and using (2.4) yields

$$\tau = \sum_X \theta_X \gamma_X^{-1} n_X.$$

Tensoring with an invertible object preserves n_X. Moreover, (2.1) and the balancing formula imply

$$\theta_{t^{-1} \otimes X} = \theta_{t^{-1}} \theta_X \gamma_X^{-1}.$$

Reindexing by $X \mapsto t^{-1} \otimes X$ therefore gives $\tau = \theta_{t^{-1}}\tau$, proving (2.2). □

In the notation of [6], the invertible object $\overline{\mathbf{1}}$ is characterized by $S_{\overline{\mathbf{1}},X}/d_{\overline{\mathbf{1}}} = d_{X^*}$. Thus $\overline{\mathbf{1}} = t^{-1}$, and (2.2) is precisely [6, Proposition 2.21]. The preceding proof records the normalization of the slope used below.

Proposition 2.2. *Let $\mathcal{A}$ be a premodular fusion category such that*

$$\mathcal{Z}_2(\mathcal{A}) = \langle f \rangle \simeq \mathrm{Rep}(\mathbb{Z}/2), \qquad \dim_{\mathcal{A}}(f) = -1.$$

Then $\mathcal{A}$ admits a minimal modular extension preserving its specified spherical structure. More precisely, every minimal nondegenerate braided extension of $\mathcal{A}$ admits a spherical structure with this restriction.

Proof. Lemma A.1 provides a minimal nondegenerate extension

$$\iota : \mathcal{A} \hookrightarrow \mathcal{N}, \qquad \mathcal{A}'_{\mathcal{N}} = \langle f \rangle.$$

Fix any such extension. The double-dual functor $F = (-)^{**}_{\mathcal{N}}$ is a braided tensor autoequivalence. The given pivotal structure defines a tensor natural isomorphism $\alpha = j_{\mathcal{A}}^{-1} : F\iota \Rightarrow \iota$. By the fixed-boundary assertion of Lemma A.1, there is a tensor natural isomorphism $\beta : F \Rightarrow \mathrm{id}_{\mathcal{N}}$ with $\beta\iota = \alpha$. Consequently

$$j_{\mathcal{N}} := \beta^{-1} : \mathrm{id}_{\mathcal{N}} \longrightarrow (-)^{**}_{\mathcal{N}}$$

is a pivotal structure restricting to the specified $j_{\mathcal{A}}$. Duals are identified along ι using its tensor structure.

The slope of $j_{\mathcal{N}}$ is trivial on $\mathcal{A}$ because $j_{\mathcal{A}}$ is spherical. Hence the object t in Lemma 2.1 belongs to $\mathcal{A}'_{\mathcal{N}} = \langle f \rangle$, so $t = \mathbf{1}$ or f. The Tannakian braiding has $c_{f,f} = \mathrm{id}$, whereas the prescribed dimension is -1. The twist–dimension relation for this invertible object gives $\theta_f = -1$. Since $f^{-1} = f$, (2.2) excludes $t = f$. Thus the slope is trivial, and $j_{\mathcal{N}}$ is spherical. □

The negative dimension in Proposition 2.2 is essential to the conclusion about every fixed extension. It is the twist $\theta_f = -1$, rather than the vanishing of an existence obstruction, that rules out a nonspherical pivotal extension.

3 Construction of the Galois–modular extension

We first record the integrality facts used in the construction. A tensor functor is *dominant* if every simple object of its target occurs in the image of some object.

Lemma 3.1. *The target of a dominant tensor functor from an integral fusion category is integral. If a finite group H acts on a fusion category $\mathcal{C}$, then $\mathcal{C}^H$ is integral if and only if $\mathcal{C}$ is integral.*

Proof. The first assertion is the fusion-category case of [3, Corollary 8.36]; we include a direct argument. Let $F : \mathcal{U} \to \mathcal{V}$ be dominant, with $\mathcal{U}$ integral. Choose $V \in \mathcal{U}$ so that $F(V)$ contains every simple object of $\mathcal{V}$. If $(Y_i)_i$ are these simple objects, the matrix

$$A_{ij} = \dim \mathrm{Hom}(Y_j, F(V) \otimes Y_i)$$

has strictly positive integer entries. Positivity follows by choosing a simple constituent of $Y_j \otimes Y_i^*$ and applying Frobenius reciprocity. The vector $d_i = \mathrm{FPdim}(Y_i)$ satisfies

$$Ad = \mathrm{FPdim}(F(V))d = \mathrm{FPdim}(V)d.$$

The eigenvalue is an integer, and its eigenspace is one-dimensional by the Perron–Frobenius theorem. This eigenspace is therefore defined over $\mathbb{Q}$. Normalizing by $d_{\mathbf{1}} = 1$ shows that every d_i is rational. Since Frobenius–Perron dimensions are algebraic integers, all d_i are integers.

The forgetful functor $\mathcal{C}^H \to \mathcal{C}$ is dominant, which proves one direction of the second assertion. Conversely, a simple equivariant object associated to a simple $Y \in \mathcal{C}$ and an irreducible projective representation π of its stabilizer H_Y has Frobenius–Perron dimension

$$[H : H_Y] \dim(\pi) \, \mathrm{FPdim}(Y).$$

This is an integer when $\mathcal{C}$ is integral. □

Proof of Theorem 1.1. By [5, Proposition 2.3], there is a nondegenerate braided extension

$$\mathcal{B} \hookrightarrow \mathcal{M}_0, \qquad \mathcal{D}_0 := \mathcal{B}'_{\mathcal{M}_0} \quad \text{integral}. \tag{3.1}$$

We first replace $\mathcal{M}_0$ by an ambient category carrying the required spherical structure.

Let $\widehat{\mathcal{M}}_0$ be the canonical sphericalization of $\mathcal{M}_0$ [2, Section 2.4.4]. Its objects are pairs

$$(X, \varphi), \qquad \varphi : X \xrightarrow{\sim} X^{**}, \qquad \varphi^{**} \circ \varphi = \delta_X,$$

where δ is the canonical Radford isomorphism. Morphisms commute with the displayed isomorphisms, the tensor product uses the tensor structure of $(-)^{**}$, and the pivotal structure is φ. This structure is spherical. The braiding is inherited from $\mathcal{M}_0$, and the forgetful tensor functor is braided and dominant. Every simple object of $\mathcal{M}_0$ has two simple lifts, with the same Frobenius–Perron dimension as the original object. In particular,

$$\operatorname{FPdim}(\widehat{\mathcal{M}}_0) = 2\operatorname{FPdim}(\mathcal{M}_0).$$

Nondegeneracy of $\mathcal{M}_0$ implies

$$\mathcal{Z}_2(\widehat{\mathcal{M}}_0) = \langle \varepsilon \rangle \simeq \operatorname{Rep}(\mathbb{Z}/2), \qquad \varepsilon = (\mathbf{1}, -\operatorname{id}), \qquad \dim(\varepsilon) = -1. \tag{3.2}$$

Indeed, the underlying object of a transparent object must be transparent in $\mathcal{M}_0$; the two lifts of the unit give exactly the indicated center.

The specified spherical structure on $\mathcal{B}$ defines a fully faithful braided spherical lift

$$X \longmapsto (X, j_{\mathcal{B},X}) \quad \text{into } \widehat{\mathcal{M}}_0.$$

Here $j_{\mathcal{B}}^{**} j_{\mathcal{B}} = \delta|_{\mathcal{B}}$: the canonical Radford isomorphism restricts to that of a full fusion subcategory, as follows from its characterization by the left and right traces on simple objects. Thus the displayed pairs satisfy the defining condition of sphericalization.

Proposition 2.2 now gives a minimal modular extension preserving this spherical structure,

$$\widehat{\mathcal{M}}_0 \hookrightarrow \mathcal{N}.$$

Set

$$\mathcal{K} = \mathcal{B}'_{\mathcal{N}}, \qquad \mathcal{E} = \mathcal{Z}_2(\mathcal{B}) \simeq \operatorname{Rep}(G).$$

Minimality means $(\widehat{\mathcal{M}}_0)'_{\mathcal{N}} = \langle \varepsilon \rangle$. The double-centralizer theorem consequently gives

$$\begin{gathered} \mathcal{Z}_2(\mathcal{K}) = \mathcal{E}, \qquad \operatorname{FPdim}(\mathcal{K}) = 4\operatorname{FPdim}(\mathcal{D}_0), \\ \varepsilon'_{\mathcal{N}} = \widehat{\mathcal{M}}_0, \qquad \varepsilon'_{\mathcal{K}} = \widehat{\mathcal{D}}_0. \end{gathered} \tag{3.3}$$

In the last equality, $\widehat{\mathcal{D}}_0$ denotes the full preimage of $\mathcal{D}_0$ under the forgetful functor. Indeed, centralizing the lifted $\mathcal{B}$ inside $\widehat{\mathcal{M}}_0$ is exactly the condition of centralizing $\mathcal{B}$ after forgetting the lift. This preimage is integral. Thus $\mathcal{K}$ is weakly integral, and the centralizer of ε in $\mathcal{K}$ is integral.

By [3, Propositions 8.23 and 8.24], $\mathcal{K}$ is pseudounitary and has a canonical positive spherical structure j_+. Write its inherited spherical structure as

$$j = j_+ \chi, \qquad \chi \in \operatorname{Aut}_{\otimes}(\operatorname{id}_{\mathcal{K}}). \tag{3.4}$$

On a simple object, $\chi_{X^*} = \chi_X^{-1}$. Since both structures are spherical and $\operatorname{FPdim}(X) = \operatorname{FPdim}(X^*)$, we also have $\chi_X = \chi_{X^*}$. Hence $\chi_X \in \{1, -1\}$. Moreover,

$$\chi_\varepsilon = -1. \tag{3.5}$$

The restriction of this sign character to $\mathcal{E}$ determines the rest of the construction.

Case 1: $\chi|_{\mathcal{E}} = 1$. De-equivariantize $(\mathcal{K}, j_+)$ by its Tannakian center to obtain the nondegenerate modular category

$$\mathcal{Q} = \mathcal{K}_G.$$

The sign character descends to $\mathcal{Q}$: it acts trivially on the regular algebra of $\mathcal{E}$, so its component on each module is an algebra-linear automorphism. Let e be the image of ε. Then e is invertible, $e^2 = \mathbf{1}$, and the descended character takes value -1 on e, so $e \neq \mathbf{1}$. By (3.3),

$$e'_{\mathcal{Q}} = (\widehat{\mathcal{D}}_0)_G.$$

This equality can be checked on underlying modules: their monodromy with the free image of ε is precisely the induced monodromy with ε. Lemma 3.1, applied to the de-equivariantization functor, shows that $e'_{\mathcal{Q}}$ is integral.

By (2.3), the descended character is monodromy with an invertible object $a \in \mathcal{Q}$. Equation (3.5) says that the monodromy of a and e is -1. Monodromy with e defines a $\mathbb{Z}/2$-grading whose even component is $e'_{\mathcal{Q}}$. Tensoring with a is a bijection between the even and odd simple objects and preserves Frobenius–Perron dimensions. Both components are therefore integral. Since $\mathcal{K} \simeq \mathcal{Q}^G$, Lemma 3.1 implies that $\mathcal{K}$ is integral. We take $\mathcal{M} = \mathcal{N}$.

Case 2: $\chi|_{\mathcal{E}} \neq 1$. Let $\mathcal{K}_+ = \ker \chi$ be the full fusion subcategory generated by the simple objects with sign $+1$, and put

$$\mathcal{E}_+ = \mathcal{E} \cap \mathcal{K}_+.$$

Both inclusions $\mathcal{K}_+ \subset \mathcal{K}$ and $\mathcal{E}_+ \subset \mathcal{E}$ have index two. The centralizer dimension formula for a possibly degenerate braided category gives

$$\mathrm{FPdim}((\mathcal{K}_+)'_{\mathcal{K}}) = \frac{\mathrm{FPdim}(\mathcal{K})\,\mathrm{FPdim}(\mathcal{E}_+)}{\mathrm{FPdim}(\mathcal{K}_+)} = \mathrm{FPdim}(\mathcal{E}).$$

As this centralizer contains $\mathcal{E}$, we conclude that

$$(\mathcal{K}_+)'_{\mathcal{K}} = \mathcal{E}, \qquad \mathcal{Z}_2(\mathcal{K}_+) = \mathcal{E}_+. \tag{3.6}$$

The inherited spherical structure on $\mathcal{K}_+$ is positive, and $\mathcal{E}_+$ is Tannakian. Applying [5, Propositions 2.3 and 2.4] to $\mathcal{K}_+$ gives a pseudounitary modular extension with its positive spherical structure,

$$\mathcal{K}_+ \hookrightarrow \mathcal{P}, \qquad \mathcal{I} := (\mathcal{K}_+)'_{\mathcal{P}} \quad \text{integral}. \tag{3.7}$$

This embedding preserves the positive spherical structure.

In $\mathcal{N} \boxtimes \mathcal{P}^{\mathrm{rev}}$, take the canonical diagonal connected étale algebra

$$L = \bigoplus_{X \in \mathrm{Irr}(\mathcal{K}_+)} X \boxtimes X^*. \tag{3.8}$$

The algebra structure is that obtained from the right adjoint of the central tensor product functor $\mathcal{K}_+ \boxtimes \mathcal{K}_+^{\mathrm{rev}} \to \mathcal{K}_+$ applied to the unit; see [1, Lemma 3.5]. The two embeddings of $\mathcal{K}_+$ have the same spherical structure. Since the twist in the second factor is reversed,

$$\theta_L = \mathrm{id}_L, \qquad \dim(L) = \sum_{X \in \mathrm{Irr}(\mathcal{K}_+)} \mathrm{FPdim}(X)^2 = \mathrm{FPdim}(\mathcal{K}_+) > 0.$$

The multiplication pairing $L \otimes L \to L \to \mathbf{1}$ is nondegenerate. To see this, choose the projection $L \to \mathbf{1}$ to split the unit. The regular right L-module is simple by connectedness and separability. The pairing induces a nonzero module morphism $L \to L^*$, hence an isomorphism.

It follows from [1, Corollary 3.30 and Remark 3.31(ii)] that the category of local modules

$$\mathcal{M} = (\mathcal{N} \boxtimes \mathcal{P}^{\mathrm{rev}})_L^0 \tag{3.9}$$

is modular, with the twist induced from the ambient category. The subcategory $\mathcal{B} \boxtimes \mathbf{1}$ centralizes L. Consequently free modules define a braided, twist-preserving functor

$$J : \mathcal{B} \longrightarrow \mathcal{M}, \qquad J(X) = (X \boxtimes \mathbf{1}) \otimes L.$$

It preserves the spherical structure because a pivotal structure is determined by the braiding and its associated twist. Moreover, the free–forgetful adjunction gives

$$\begin{aligned} \mathrm{Hom}_L(J(X), J(Y)) &\simeq \mathrm{Hom}(X \boxtimes \mathbf{1}, (Y \boxtimes \mathbf{1}) \otimes L) \\ &\simeq \mathrm{Hom}_{\mathcal{B}}(X, Y). \end{aligned}$$

Only the summand $\mathbf{1} \boxtimes \mathbf{1}$ of L contributes in the second Deligne factor. Thus J is fully faithful.

Let $\mathcal{D} = J(\mathcal{B})'_{\mathcal{M}}$. Free modules likewise give fully faithful embeddings of $\mathcal{E}$ from the first factor and of $\mathcal{I}^{\mathrm{rev}}$ from the second factor into $\mathcal{D}$. Their images centralize one another. We use the same notation for these images and now identify their intersection. For simple $U \in \mathcal{E}$ and $V \in \mathcal{I}$, adjunction gives

$$\begin{aligned} &\mathrm{Hom}_L((U \boxtimes \mathbf{1}) \otimes L, (\mathbf{1} \boxtimes V) \otimes L) \\ &\quad \simeq \bigoplus_{X \in \mathrm{Irr}(\mathcal{K}_+)} \mathrm{Hom}_{\mathcal{N}}(U, X) \otimes \mathrm{Hom}_{\mathcal{P}}(\mathbf{1}, V \otimes X^*). \end{aligned}$$

This space is nonzero exactly when U is in $\mathcal{E}_+$ and V is its image under (3.7). In that case it is one-dimensional. Thus

$$\mathcal{E} \cap \mathcal{I}^{\mathrm{rev}} = \mathcal{E}_+.$$

Writing $\vee$ for the fusion subcategory generated by two subcategories, the dimension formula for their join yields

$$\mathrm{FPdim}(\mathcal{E} \vee \mathcal{I}^{\mathrm{rev}}) = \frac{\mathrm{FPdim}(\mathcal{E})\, \mathrm{FPdim}(\mathcal{I})}{\mathrm{FPdim}(\mathcal{E}_+)} = 2\, \mathrm{FPdim}(\mathcal{I}). \tag{3.10}$$

On the other hand, [1, Corollary 3.32] and the centralizer dimension formula give

$$\begin{aligned} \mathrm{FPdim}(\mathcal{D}) &= \frac{\mathrm{FPdim}(\mathcal{N})\, \mathrm{FPdim}(\mathcal{P})}{\mathrm{FPdim}(\mathcal{B})\, \mathrm{FPdim}(\mathcal{K}_+)^2} \\ &= \frac{\mathrm{FPdim}(\mathcal{K})\, \mathrm{FPdim}(\mathcal{P})}{\mathrm{FPdim}(\mathcal{K}_+)^2} = 2\, \mathrm{FPdim}(\mathcal{I}). \end{aligned}$$

Comparison with (3.10) proves

$$\mathcal{D} = \mathcal{E} \vee \mathcal{I}^{\mathrm{rev}}.$$

Because the two generating subcategories centralize one another, tensor product defines a dominant tensor functor

$$\mathcal{E} \boxtimes \mathcal{I}^{\mathrm{rev}} \longrightarrow \mathcal{D}.$$

Its source is integral, so Lemma 3.1 makes $\mathcal{D}$ integral. This completes the construction in the second case, and hence the proof. □

References

[1] A. Davydov, M. Müger, D. Nikshych, and V. Ostrik. The Witt group of non-degenerate braided fusion categories. *J. Reine Angew. Math.* **677** (2013), 135–177. doi:10.1515/crelle.2012.014. arXiv:1009.2117.

[2] V. Drinfeld, S. Gelaki, D. Nikshych, and V. Ostrik. On braided fusion categories I. *Selecta Math. (N.S.)* **16** (2010), 1–119. arXiv:0906.0620.

[3] P. Etingof, D. Nikshych, and V. Ostrik. On fusion categories. *Ann. of Math. (2)* **162** (2005), 581–642. arXiv:math/0203060.

[4] P. Etingof, D. Nikshych, and V. Ostrik. Fusion categories and homotopy theory. *Quantum Topol.* **1** (2010), 209–273. arXiv:0909.3140.

[5] T. Johnson–Freyd. Schopieray's Galois-modular extension conjecture. arXiv:2601.23192 (2026).

[6] A. Lacabanne. Slightly degenerate categories and $\mathbb{Z}$-modular data. *Int. Math. Res. Not. IMRN* **2021**, no. 12, 9340–9374. doi:10.1093/imrn/rnz105. arXiv:1807.00766. Proposition numbers cited here refer to the published version.

[7] M. Müger. On the structure of modular categories. *Proc. London Math. Soc. (3)* **87** (2003), 291–308. arXiv:math/0201017.

[8] J. Plavnik, A. Schopieray, Z. Yu, and Q. Zhang. Modular tensor categories, subcategories, and Galois orbits. *Transform. Groups* **29** (2024), 1623–1648. doi:10.1007/s00031-022-09787-9. arXiv:2111.05228.

A Extending a pivotal structure with fixed boundary

The pivotal extension argument needs a natural isomorphism that agrees with the prescribed one on the subcategory. The following formulation keeps that boundary identification as part of the data.

Lemma A.1. *Let $\mathcal{A}$ be a braided fusion category with $\mathcal{Z}_2(\mathcal{A}) \simeq \mathrm{Rep}(\mathbb{Z}/2)$. There is a minimal nondegenerate braided extension of $\mathcal{A}$. For any such extension $\iota : \mathcal{A} \hookrightarrow \mathcal{N}$, any braided tensor autoequivalence F of $\mathcal{N}$, and any specified tensor natural isomorphism $\alpha : F\iota \Rightarrow \iota$, there is a tensor natural isomorphism*

$$\beta : F \Rightarrow \mathrm{id}_{\mathcal{N}} \qquad \textit{such that} \qquad \beta\iota = \alpha.$$

Proof. Write $H = \mathbb{Z}/2$, fix the Tannakian identification, and let $R = \mathrm{Fun}(H) \in \mathrm{Rep}(H)$ be the regular algebra. De-equivariantization gives a nondegenerate braided fusion category $\mathcal{C} = \mathcal{A}_H$ with a specified categorical H-action and an equivalence $\mathcal{A} \simeq \mathcal{C}^H$. Minimal nondegenerate extensions of $\mathcal{A}$ correspond to faithfully graded braided H-crossed extensions of $\mathcal{C}$ realizing this action. This correspondence is compatible with functors and tensor natural isomorphisms, by the 2-equivalence of [2, Theorem 4.44].

In particular, a boundary-preserving autoequivalence is a pair (F, α) as in the statement, not just F with an unspecified identification of its restriction. The isomorphism α_R identifies $F(\iota(R))$ with $\iota(R)$ and hence induces a functor on the module categories used in de-equivariantization. The remaining components of α retain its specified identification on $\mathcal{C}$ and the categorical H-action. Equivariantization recovers the pair (F, α). Under this correspondence, a morphism $(F, \alpha) \to (F', \alpha')$ is a tensor natural isomorphism $\beta : F \Rightarrow F'$ satisfying

$$\alpha' \circ (\beta\iota) = \alpha. \tag{A.1}$$

We use based classifying maps, retaining the identification of the neutral category. We now apply the homotopy description of extension theory [4, Theorem 5.2, Sections 7.1–7.2, and Theorem 7.12]. Let $X = B\mathbf{Pic}(\mathcal{C})$ be the classifying space of the Picard 2-group of invertible module categories, and let $Y = B\mathbf{Aut}^{\mathrm{br}}(\mathcal{C})$ be the classifying space of the categorical group of braided autoequivalences. The natural map $X \to Y$ induces isomorphisms on π_1 and π_2, and Y has no higher homotopy groups. More explicitly,

$$\begin{aligned} \pi_2(X) &= \mathrm{Inv}(\mathcal{C}), \qquad & \pi_3(X) &= \mathbb{C}^\times, \\ \pi_2(Y) &= \mathrm{Aut}_\otimes(\mathrm{id}_{\mathcal{C}}), \qquad & \pi_i(Y) &= 0 \quad (i \geq 3). \end{aligned}$$

These are [4, Propositions 7.3 and 7.5]; the map on π_2 is the monodromy isomorphism (2.3). Thus the homotopy fiber of $X \to Y$ is $K(\mathbb{C}^\times, 3)$. The coefficient action is trivial, since all functors are $\mathbb{C}$-linear.

The specified categorical action is a map $BH \to Y$. The extension problem is to lift this map to X, including the chosen identification of its image in Y with that fixed action. Choose a normalized cocycle $\omega \in Z^4(H, \mathbb{C}^\times)$ representing the resulting obstruction. The cochain model for the 2-groupoid of lifts is as follows: a lift is a normalized $\psi \in C^3(H, \mathbb{C}^\times)$ with $d\psi = \omega$; an equivalence from ψ to ψ' is a normalized degree-two cochain λ with $d\lambda = \psi'/\psi$; and a natural isomorphism from λ to λ' is a normalized degree-one cochain μ with $d\mu = \lambda'/\lambda$. Here C^r and Z^r denote group cochains and cocycles, and d is the group-cohomology coboundary. The successive coherence conditions are exactly these coboundary equations in the nerve model of [4, Section 7.1].

It follows that existence is obstructed only by $[\omega] \in H^4(H, \mathbb{C}^\times)$, while, at any chosen lift, isomorphism classes of boundary-preserving autoequivalences form

$$H^2(H, \mathbb{C}^\times). \tag{A.2}$$

The fixed identification with the action is essential here: the natural isomorphisms in this homotopy fiber obey (A.1).

For $H = \mathbb{Z}/2$ and the trivial coefficient action, the cyclic resolution gives

$$H^{2r}(H, \mathbb{C}^\times) = \mathbb{C}^\times/(\mathbb{C}^\times)^2 = 0 \qquad (r \geq 1).$$

Hence the degree-four obstruction vanishes, giving an extension, and (A.2) is trivial for every extension. Explicitly, a normalized degree-two cocycle is determined by its value $\lambda(s,s)$ at the nontrivial element $s \in H$. Choose $\mu(s) \in \mathbb{C}^\times$ with $\mu(s)^2 = \lambda(s,s)$ and $\mu(1) = 1$; then $d\mu = \lambda$. Every pair (F, α) is consequently isomorphic to $(\mathrm{id}_{\mathcal{N}}, \mathrm{id}_\iota)$. Equation (A.1) gives precisely $\beta\iota = \alpha$, as required. □

B A pointed obstruction in a fixed extension

Let $\mathcal{T}$ be the pointed modular category with group

$$A = \mathbb{Z}/4 \times \mathbb{Z}/4$$

and quadratic form $q(a,b) = i^{ab}$, equipped with its positive spherical structure. Its monodromy pairing is

$$m((a,b),(c,d)) = i^{ad+bc},$$

which is nondegenerate. Take $f = (2,0)$ and set $\mathcal{B} = \langle f \rangle'_{\mathcal{T}}$. Then

$$\mathrm{Irr}(\mathcal{B}) = \{(a,b) \in A : b \text{ is even}\}, \qquad \mathcal{Z}_2(\mathcal{B}) = \langle f \rangle \simeq \mathrm{Rep}(\mathbb{Z}/2).$$

In particular, $\mathcal{T}$ is a minimal nondegenerate extension of $\mathcal{B}$, and f has trivial self-braiding.

Modify the positive spherical structure on $\mathcal{B}$ by the sign character

$$\chi(a,b) = i^b \in \{1, -1\}.$$

This character takes value $+1$ on f, but it does not extend to a spherical structure on $\mathcal{T}$. Indeed, spherical structures on this pointed category differ from the positive one by characters $A \to \{1, -1\}$. Every such character is trivial on $2A$ and hence on $(0,2)$, whereas $\chi(0,2) = -1$.

The character $\widetilde{\chi}(a,b) = i^b$ on all of A does give a pivotal extension. Its slope is

$$\widetilde{\chi}(a,b)^2 = (-1)^b = m(f,(a,b)),$$

so it is nonspherical. The twist of f remains $+1$, consistently with Lemma 2.1. This is exactly the possibility excluded by the hypothesis $\dim(f) = -1$ in Proposition 2.2. Theorem 1.1 permits a different ambient modular category and therefore does not require the prescribed structure to extend within $\mathcal{T}$.