

Copies of $K_{4,4}$ in $K_{3,5}$ -free graphs

Abstract

We prove that the maximum number of copies of $K_{4,4}$ in an n -vertex $K_{3,5}$ -free graph is $\Theta(n^3)$, with the lower bound attained by bipartite graphs. Together with a theorem of Wang, Yang, and Zhou (arXiv 2026), this establishes the cubic order for $K_{t,t}$ copies in $K_{3,t+1}$ -free graphs for every fixed $t \geq 4$, answering a question of Pohoata, Tidor, and Yu (arXiv 2026). Our construction uses the degree-four specialization of Wang, Yang, and Zhou's finite-field point set. Large plane sections are unavoidable at this degree, and reducing all sections to at most four points by point deletion retains only a vanishing proportion of the point set. We instead alter the incidence graph obtained from a random projective pairing. Preliminary edge deletions limit how many exceptional complete bipartite subgraphs contain any remaining edge. A geometric intersection bound then permits local random pruning that eliminates every $K_{3,5}$ while preserving a constant proportion of the counted $K_{4,4}$'s.

Note. This paper was generated entirely by AI, including MiMo, using an automated research pipeline developed by Chenghua Liu and Hanyu Li.

1 Introduction

A $K_{3,5}$ -free graph has at most four common neighbors for every triple of vertices. A copy of $K_{4,4}$ saturates this restriction on every triple in either of its parts. In fact, a triple can lie in a part of at most one such copy, which gives a cubic upper bound on the total number of copies. Attaining this bound requires many complete bipartite subgraphs to coexist without allowing even one triple a fifth common neighbor.

For finite simple graphs H and F , write $\text{ex}(n, H, F)$ for the maximum number of unlabeled, not necessarily induced copies of H in an n -vertex F -free graph. The problem above is the smallest instance of the question of Pohoata, Tidor, and Yu [2, Question 4.1]: for every fixed $3 \leq s < t$, is

$$\text{ex}(n, K_{t,t}, K_{s,t+1}) = \Theta_{s,t}(n^s)? \quad (1)$$

They proved the corresponding statement for $s = 2$. We establish the case $(s, t) = (3, 4)$.

Theorem 1. *There is an absolute constant $c > 0$ such that, for every sufficiently large integer n , some bipartite n -vertex $K_{3,5}$ -free graph contains at least cn^3 copies of $K_{4,4}$. Consequently,*

$$\text{ex}(n, K_{4,4}, K_{3,5}) = \Theta(n^3).$$

The missing parameter is the size of the forbidden common neighborhood. The general lower bound of Gerbner and Patkós [1, Theorem 1.6(iii)] gives $\Omega(n^{3/2-o(1)})$ copies at our parameters. Janzer, Longbrake, and Yepremyan [3, Theorem 1.1] obtained the cubic order for $\text{ex}(n, K_{a,b}, K_{3,t})$, for fixed $3 < a \leq b$ and sufficiently large t . Hou, Hu, and Wang [4, Theorem 1.2] made the sufficient threshold explicit:

$$t \geq 2 \max\{3, \lceil b/2 \rceil\} + 1.$$

For $a = b = 4$, this guarantees many $K_{4,4}$'s while excluding $K_{3,7}$; it does not exclude $K_{3,5}$. Wang, Yang, and Zhou [5, Theorem 1.1] reached $t = b + 1$ for $b \geq 5$ and $4 \leq a \leq b$, leaving $b = 4$ outside their range. Thus Theorem 1, together with their theorem, answers (1) for every $s = 3$ and $t \geq 4$.

The finite-geometric approach explains both the cubic scale and the obstruction at this last parameter. A cap in $\text{PG}(5, q)$ is a set of points with no three collinear, so every triple spans a plane. In a point–hyperplane incidence graph, its common neighbors are then described by a plane section of the point set. Four-point sections can produce $K_{4,4}$ ’s, whereas larger sections can produce forbidden common neighborhoods. A set with q^3 points and a positive proportion of the $\Theta(q^9)$ planes meeting it in four points has the appropriate scale for a cubic construction.

We use the point set of Wang, Yang, and Zhou [5, Section 3], specializing its defining polynomial f to degree four. On a typical plane, the section is given by the roots of $f + A^2 - \alpha B^2 - C$, where α is a fixed nonsquare and A, B, C have degree at most two. When $\deg f \geq 5$, the other terms cannot cancel its leading coefficient. At degree four, the entire section polynomial can vanish. This happens for $q^3(q+1)$ planes for every monic quartic, so choosing f randomly cannot eliminate these sections. Moreover, every point belongs to $q(q+1)$ of these identity planes. A subset meeting each of them in at most four points therefore has size at most $4q^2$. Point deletion would lose the q^3 scale on which the construction depends. The simultaneous-splitting argument in [5, Section 4] also requires a shift of degree strictly less than $\deg f$, whereas the shift can have degree four here.

We retain all points and modify the incidence graph. First choose f so that there are $\Omega(q^9)$ four-point sections, then use the random projective pairing of Janzer, Longbrake, and Yepremyan [3, Section 4] to obtain $\Omega(q^9)$ designated copies in expectation. The exceptional plane sections define complete bipartite blocks containing every possible forbidden common neighborhood. Deleting all their edges is too expensive for the available counting argument: the expected number of edge–block incidences is $O(q^5)$, and an edge can belong to $O(q^4)$ designated copies. The resulting $O(q^9)$ loss has the same order as the entire supply of copies.

Instead, we first delete the blocks whose two sides come from exceptional planes. Their expected edge count is only $O(q)$, so their deletion costs $O(q^5)$ copies. We then delete edges belonging to more than L exceptional blocks, at an expected cost of $O(q^9/L)$ copies. Choosing L as a sufficiently large absolute constant leaves $\Omega(q^9)$ copies for a suitable projective pairing. Each remaining block has one side of size at most four, and each remaining edge is subject to at most L blocks.

The last step uses the cap property a second time. A designated copy meets the side of size at most four in any remaining block in at most two vertices. Three points would determine the controlled-side plane and force the opposite rich plane to be exceptional. For each vertex on the other side, we independently retain two neighbors in this side. This prevents every triple there from having a common neighbor through the block. At the same time, each choice has a positive constant probability of retaining all edges of a fixed designated copy that it affects. There are at most $16L$ affecting choices per copy, so its survival probability stays bounded away from zero as q grows. This is why controlling block overlap before pruning preserves the cubic order. The geometry is established in Section 2, and the two alteration stages are proved in Sections 3 and 4.

For completeness, we give the upper-bound count used above. Write $N(H, G)$ for the number of copies of H in G . For a vertex set X , let $\Gamma_G(X) := \bigcap_{x \in X} N_G(x)$ be its common neighborhood.

Lemma 2 (Upper bound). *Every n -vertex $K_{3,5}$ -free graph G satisfies*

$$N(K_{4,4}, G) \leq \frac{1}{8} \binom{n}{3}.$$

Proof. Fix a triple $X \subseteq V(G)$ that is contained in one part of a copy of $K_{4,4}$. The opposite part must be $Y = \Gamma_G(X)$, since $|Y| = 4$ and $|\Gamma_G(X)| \leq 4$. For any triple $Y' \subseteq Y$,

$$X \subseteq \Gamma_G(Y) \subseteq \Gamma_G(Y'), \quad |\Gamma_G(Y')| \leq 4.$$

There is therefore at most one possible fourth vertex in the part containing X . Hence X belongs to at most one copy as a triple within a part. Each copy has four such triples in each of its two parts. Counting these eight triples per copy proves the bound. $\square$

It remains to prove the lower bound. Throughout the construction, all implied constants are absolute, and limits in q are taken through odd prime powers.

2 A quartic cap with many four-point sections

The choice of the quartic must produce many four-point sections, but it cannot avoid long sections. We first classify the latter and count their incidences for an arbitrary quartic. Only then do we choose its coefficients to obtain the four-point sections used for the copies.

Let $q > 4$ be an odd prime power, and let $\mathbb{F}_q$ be the field with q elements. The projective space $\text{PG}(5, q)$ consists of the one-dimensional subspaces of $\mathbb{F}_q^6$; a projective plane corresponds to a three-dimensional vector subspace. Subspace dimensions below are projective unless explicitly described as affine. A *cap* is a set with no three collinear points.

Fix a nonsquare $\alpha \in \mathbb{F}_q$ and a monic quartic

$$f(t) = t^4 + f_3 t^3 + f_2 t^2 + f_1 t + f_0 \in \mathbb{F}_q[t].$$

In homogeneous coordinates $[w : x : y : z : u : v]$, set

$$S = S_f := \{[1 : x : y : z : x^2 : y^2 - \alpha z^2 + f(x)] : x, y, z \in \mathbb{F}_q\}. \quad (2)$$

Thus $|S| = q^3$. This is the quartic specialization of the point set in [5, Section 3]; we give all the section arguments needed here.

Lemma 3 (Cap property). *No three points of S are collinear.*

Proof. Suppose that three distinct points of S are collinear. Their (x, y, z) -coordinates can be written as

$$p, \quad p + h, \quad p + \lambda h, \quad h \in \mathbb{F}_q^3 \setminus \{0\}, \quad \lambda \in \mathbb{F}_q \setminus \{0, 1\}.$$

The affine collinearity relation in the $u = x^2$ coordinate gives

$$\lambda(\lambda - 1)h_x^2 = 0,$$

so $h_x = 0$. The same relation in the $v = y^2 - \alpha z^2 + f(x)$ coordinate now gives

$$\lambda(\lambda - 1)(h_y^2 - \alpha h_z^2) = 0.$$

Since α is a nonsquare, $h_y^2 - \alpha h_z^2 = 0$ implies $h_y = h_z = 0$. This contradicts $h \neq 0$. $\square$

Every triple now determines a unique plane. To understand how many other points that plane can contain, consider the affine projection

$$\rho : \mathbb{F}_q^5 \longrightarrow \mathbb{F}_q^2, \quad (x, y, z, u, v) \longmapsto (x, u).$$

For a projective plane Π not contained in $w = 0$, let $H = \Pi \cap \{w = 1\}$. If $\dim \rho(H) = 2$, the restriction of ρ to H is an affine isomorphism, so H has a unique representation

$$y = A(x, u), \quad z = B(x, u), \quad v = C(x, u), \quad (3)$$

where A, B, C are affine functions of (x, u) . We call Π a *graph plane*. Upon substituting $u = t^2$, write

$$A(t) = a_0 + a_1t + a_2t^2, \quad B(t) = b_0 + b_1t + b_2t^2, \quad C(t) = c_0 + c_1t + c_2t^2.$$

The points of $S \cap \Pi$ correspond bijectively to the roots in $\mathbb{F}_q$ of

$$P_\Pi(t) = f(t) + A(t)^2 - \alpha B(t)^2 - C(t), \quad (4)$$

a polynomial of degree at most four.

Let $\mathcal{I}$ be the family of graph planes satisfying $P_\Pi \equiv 0$, called *identity planes*. For each $c \in \mathbb{F}_q$, put

$$F_c := \{x = cw, u = c^2w\} \cong \text{PG}(3, q),$$

and let $\mathcal{F}$ be the family of all projective planes contained in at least one F_c . Define the exceptional family and its section sizes by

$$\mathcal{D} := \mathcal{I} \cup \mathcal{F}, \quad a_E := |S \cap E| \quad (E \in \mathcal{D}). \quad (5)$$

Membership in $\mathcal{D}$ does not require a section to have more than four points; in particular, some exceptional planes miss S .

The exceptional family is defined by the form of the plane rather than a threshold on its section size. This makes its incidences exactly countable and ensures that every section larger than four is included.

Proposition 4 (Plane sections and exceptional incidences). *Every plane $\Pi \notin \mathcal{D}$ satisfies $|S \cap \Pi| \leq 4$. The families $\mathcal{I}$ and $\mathcal{F}$ are disjoint, and*

$$|\mathcal{I}| = q^3(q+1), \quad (6)$$

$$|\mathcal{F}| = q^4 + q^3 + q^2 + 1, \quad (7)$$

$$\sum_{E \in \mathcal{D}} a_E = 2q^5 + 2q^4 + q^3, \quad (8)$$

$$\max_{E \in \mathcal{D}} a_E \leq 2q. \quad (9)$$

Each identity plane contains exactly q points of S , and each point of S lies in exactly $q(q+1)$ identity planes.

Proof. We first prove the bound outside $\mathcal{D}$. A plane contained in $w = 0$ misses S . Otherwise, use the affine plane H defined above. If $\dim \rho(H) = 2$ and $\Pi \notin \mathcal{I}$, the nonzero polynomial (4) has at most four roots.

If $\dim \rho(H) = 1$, the affine line $\rho(H)$ meets the parabola $u = x^2$ in at most two points: substituting (t, t^2) into a nonzero affine-linear equation of that line gives a nonzero polynomial of degree at most two. Above each intersection point, the fiber in H is an affine line. By Lemma 3, it contains at most two points of S . Thus $|S \cap \Pi| \leq 4$. If $\dim \rho(H) = 0$ and $H \cap S \neq \emptyset$, then $\rho(H) = \{(c, c^2)\}$ for some c , and the projective closure of H lies in F_c . Hence $\Pi \in \mathcal{F}$. These cases prove the section bound outside $\mathcal{D}$.

We next count the identity planes. For a graph plane to belong to $\mathcal{I}$, its quartic and cubic coefficients must satisfy

$$1 + a_2^2 - \alpha b_2^2 = 0, \quad (10)$$

$$f_3 + 2a_2a_1 - 2\alpha b_2b_1 = 0. \quad (11)$$

In $\mathbb{F}_{q^2} = \mathbb{F}_q(\sqrt{\alpha})$, equation (10) is the norm equation

$$N_{\mathbb{F}_{q^2}/\mathbb{F}_q}(a_2 + b_2\sqrt{\alpha}) = -1.$$

The norm is $z \mapsto z^{q+1}$. Since $\mathbb{F}_{q^2}^\times$ is cyclic of order $q^2 - 1$, its norm map has kernel of order $q + 1$ and image $\mathbb{F}_q^\times$. Consequently, (10) has exactly $q + 1$ solutions (a_2, b_2) , all nonzero. For each such pair, (11) is a nonzero affine-linear equation in (a_1, b_1) , with exactly q solutions. The coefficients a_0, b_0 are arbitrary, after which $C = f + A^2 - \alpha B^2$ is uniquely determined and has degree at most two. Therefore

$$|\mathcal{I}| = (q + 1)q q^2 = q^3(q + 1).$$

An identity plane contains one point of S for each $x \in \mathbb{F}_q$, so its section size is q . For a fixed point $[1 : x_0 : y_0 : z_0 : x_0^2 : v_0] \in S$, the equations $A(x_0) = y_0$ and $B(x_0) = z_0$ determine a_0, b_0 after the other coefficients have been chosen. Thus the point belongs to exactly $q(q + 1)$ identity planes.

For the fiber family $\mathcal{F}$, each $F_c \cong \text{PG}(3, q)$ contains $q^3 + q^2 + q + 1$ planes. For $c \neq d$, the equations of $F_c \cap F_d$ imply $w = x = u = 0$. Hence distinct F_c 's meet in the same plane

$$W_\infty := \{w = x = u = 0\},$$

which is disjoint from S . It is the only plane counted in more than one F_c , and so

$$|\mathcal{F}| = 1 + q(q^3 + q^2 + q) = q^4 + q^3 + q^2 + 1.$$

There are q^2 points of S in each F_c , and each lies in $q^2 + q + 1$ planes of F_c . Every point of S belongs to exactly one F_c . Thus

$$\sum_{E \in \mathcal{F}} a_E = q^3(q^2 + q + 1).$$

The counts of planes in $\text{PG}(3, q)$ and through a fixed point follow by counting nonzero linear forms up to scalar, on $\mathbb{F}_q^4$ and on its quotient by that point, respectively.

To combine these counts, observe that an identity plane has a two-dimensional affine projection under ρ . A plane in $\mathcal{F}$ other than W_∞ has a zero-dimensional affine projection, and W_∞ has no affine points. Hence $\mathcal{I} \cap \mathcal{F} = \emptyset$, giving

$$\sum_{E \in \mathcal{D}} a_E = q q^3(q + 1) + q^3(q^2 + q + 1) = 2q^5 + 2q^4 + q^3.$$

Finally, a plane $E \subset F_c$ other than W_∞ has, in the affine chart, an equation

$$\beta v + \gamma y + \delta z + \varepsilon = 0, \quad (\beta, \gamma, \delta) \neq (0, 0, 0).$$

On S , this becomes

$$\beta(y^2 - \alpha z^2 + f(c)) + \gamma y + \delta z + \varepsilon = 0.$$

If $\beta \neq 0$, then for each z there are at most two solutions for y , giving at most $2q$ points. If $\beta = 0$, the equation is a nonconstant affine-linear equation in (y, z) , giving exactly q points. Together with the section sizes of W_∞ and the identity planes, this proves (9). $\square$

Remark 5 (The obstruction to point deletion). Suppose that $X \subseteq S$ satisfies $|X \cap E| \leq 4$ for every $E \in \mathcal{I}$. Counting incidences with identity planes gives

$$|X|q(q + 1) = \sum_{E \in \mathcal{I}} |X \cap E| \leq 4|\mathcal{I}| = 4q^3(q + 1),$$

and hence $|X| \leq 4q^2$. Thus imposing a four-point bound on all identity-plane sections by deleting points cannot retain a subset of size $\Theta(q^3)$.

The incidence count above holds for every monic quartic. We now choose one with $\Omega(q^9)$ four-point sections. At this stage only the section on one plane matters; the later random pairing will match it with a section on the other side of the graph. Thus a direct average over the four lower coefficients of f suffices.

Lemma 6 (Many four-point sections). *There is an absolute constant $c_0 > 0$ such that, for every sufficiently large odd prime power q , some monic quartic f has at least $c_0 q^9$ planes $\Pi \notin \mathcal{D}$ satisfying $|S_f \cap \Pi| = 4$.*

Proof. There are q^9 graph planes, one for each ordered triple of affine functions in (3). The leading coefficient of their section polynomial is

$$\ell_\Pi = 1 + a_2^2 - ab_2^2.$$

By the norm count above, exactly $(q+1)q^7$ graph planes have $\ell_\Pi = 0$.

Choose f_0, f_1, f_2, f_3 independently and uniformly from $\mathbb{F}_q$. For any fixed graph plane with $\ell_\Pi \neq 0$, the four lower coefficients of P_Π are uniform in $\mathbb{F}_q^4$. There are exactly $\binom{q}{4}$ polynomials with leading coefficient ℓ_Π and four distinct roots in $\mathbb{F}_q$, namely

$$\ell_\Pi \prod_{r \in R} (t - r), \quad R \in \binom{\mathbb{F}_q}{4}.$$

Here $\binom{\mathbb{F}_q}{4}$ denotes the family of four-element subsets of $\mathbb{F}_q$. The expected number of graph planes with $\ell_\Pi \neq 0$ and $|S_f \cap \Pi| = 4$ is therefore

$$(q^9 - (q+1)q^7) \frac{\binom{q}{4}}{q^4} = \left(\frac{1}{24} + o(1) \right) q^9. \quad (12)$$

Some choice of f attains at least this expectation. Each counted plane lies outside $\mathcal{I}$, since its leading coefficient is nonzero, and outside $\mathcal{F}$, since it is a graph plane. $\square$

Fix such a quartic f , and define

$$\mathcal{B} := \{S \cap \Pi : \Pi \notin \mathcal{D}, |S \cap \Pi| = 4\}.$$

For $A \in \mathcal{B}$, let Π_A be its associated plane, called its *rich plane*. This plane is unique: any three points of A are noncollinear by Lemma 3 and therefore span Π_A . Distinct rich planes give distinct four-sets, so

$$|\mathcal{B}| \geq c_0 q^9. \quad (13)$$

3 The incidence graph and preliminary alteration

Fix the cap and the family $\mathcal{B}$ from Section 2. We seek a projective pairing with many designated copies after two deletions: first remove blocks with two exceptional sides, then remove edges lying in too many blocks. To make these requirements hold for the same pairing, we will compare the copy count and deletion costs pointwise before taking expectations.

We use the point–hyperplane incidence framework of [3, Section 4]. Fix a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{F}_q^6$. For a projective subspace U , let $U^\perp$ be the projectivization of the orthogonal complement of its corresponding vector subspace. This operation reverses containment, maps planes to planes, and satisfies

$$U^{\perp\perp} = U. \quad (14)$$

Choose T uniformly from $\text{PGL}(6, q)$, the group of invertible linear transformations of $\mathbb{F}_q^6$ modulo nonzero scalar multiples. The transformation T need not preserve the form.

On two disjoint copies S_L, S_R of S , define a bipartite graph G_T^0 by

$$p_L r_R \in E(G_T^0) \iff \langle p, T(r) \rangle = 0. \quad (15)$$

The orthogonality test is independent of the nonzero vector representatives of the projective points and of the linear representative of T .

For $A, C \in \mathcal{B}$, call (A, C) a *designated pair* when

$$\Pi_A = T(\Pi_C)^\perp. \quad (16)$$

Then $T(\Pi_C) = \Pi_A^\perp$, so every point of A is orthogonal to every point of $T(C)$. Hence A_L and C_R form a $K_{4,4}$, which we call a *designated copy*. Distinct designated pairs give distinct copies, since their vertex sets in the fixed bipartition differ.

For integers $0 \leq k \leq m$, write

$$\begin{bmatrix} m \\ k \end{bmatrix}_q := \prod_{i=0}^{k-1} \frac{q^{m-i} - 1}{q^{k-i} - 1}$$

for the number of k -dimensional vector subspaces of $\mathbb{F}_q^m$, with the empty product equal to one. The formula follows by counting ordered linearly independent k -tuples and dividing by the number of ordered bases of $\mathbb{F}_q^k$. In particular, the number of projective planes in $\text{PG}(5, q)$ is

$$N_2 := \begin{bmatrix} 6 \\ 3 \end{bmatrix}_q = \frac{(q^6 - 1)(q^5 - 1)(q^4 - 1)}{(q^3 - 1)(q^2 - 1)(q - 1)} = (1 + o(1))q^9. \quad (17)$$

The group $\text{PGL}(6, q)$ acts transitively on planes: a linear isomorphism between their vector subspaces extends to an isomorphism of $\mathbb{F}_q^6$. Thus, for each fixed $A, C \in \mathcal{B}$,

$$\Pr(\Pi_A = T(\Pi_C)^\perp) = \frac{1}{N_2}.$$

If $K_0(T)$ is the number of designated copies, then

$$\mathbb{E}K_0(T) = \frac{|\mathcal{B}|^2}{N_2} = \Omega(q^9). \quad (18)$$

This gives the required number of copies on average, but G_T^0 need not be $K_{3,5}$ -free. We next isolate the blocks in which the common-neighborhood condition can fail. The estimates of their sizes will be combined with (18), rather than imposing separate high-probability events.

A triple can have more than four common neighbors in G_T^0 only when the corresponding plane section is exceptional. For $E \in \mathcal{D}$, define two complete bipartite subgraphs:

$$B_E^{\text{I}} := (S \cap T(E)^\perp)_L \times (S \cap E)_R, \quad (19)$$

$$B_E^{\text{II}} := (S \cap E)_L \times (S \cap T^{-1}(E^\perp))_R. \quad (20)$$

Here a product of two labeled vertex sets denotes the complete bipartite graph between them. Both graphs lie in G_T^0 , by (15). We call them *exceptional blocks* and keep them indexed by (I, E) or (II, E), even when their edge sets coincide.

In type I, call the left side the *controlled side* and the right side the *exceptional side*; in type II, reverse these roles. Thus the exceptional side always comes from $S \cap E$. A block is *mutual* if its

controlled-side plane is also in $\mathcal{D}$: this means $T(E)^\perp \in \mathcal{D}$ in type I and $T^{-1}(E^\perp) \in \mathcal{D}$ in type II. By Proposition 4, the controlled side of every nonmutual block has size at most four.

A uniformly random projective plane contains a fixed point of $\text{PG}(5, q)$ with probability

$$\frac{q^2 + q + 1}{q^5 + q^4 + q^3 + q^2 + q + 1} = \frac{1}{q^3 + 1}. \quad (21)$$

Indeed, count point–plane incidences and use transitivity on points. Its expected intersection size with S is therefore $q^3/(q^3 + 1)$.

Let $Z(T)$ be the sum of the edge counts of all exceptional blocks, including their indexing multiplicity. For each fixed E , both controlled-side planes in (19) and (20) are uniformly distributed over all projective planes. Hence

$$\mathbb{E}Z(T) = \frac{2q^3}{q^3 + 1} \sum_{E \in \mathcal{D}} a_E = O(q^5). \quad (22)$$

Let $U(T)$ be the corresponding sum over mutual blocks only. For fixed $E, F \in \mathcal{D}$, each event $T(E)^\perp = F$ or $T^{-1}(E^\perp) = F$ has probability $1/N_2$, and the block then has $a_E a_F$ edges. Therefore

$$\mathbb{E}U(T) = \frac{2}{N_2} \left(\sum_{E \in \mathcal{D}} a_E \right)^2 = O(q). \quad (23)$$

Here we used (8) and (17). To turn these edge estimates into bounds on lost copies, we need a uniform bound on the number of designated copies containing one edge.

Lemma 7 (Copies through an edge). *Every edge of G_T^0 belongs to at most*

$$M_q := \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = O(q^4)$$

designated copies.

Proof. If an edge $p_L r_R$ belongs to the designated copy (A, C) , then

$$p \in \Pi_A \subseteq T(r)^\perp.$$

The planes through p in the hyperplane $T(r)^\perp \cong \text{PG}(4, q)$ correspond to two-dimensional subspaces of the four-dimensional vector quotient by p . There are $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q$ such planes. Once Π_A is chosen, $A = S \cap \Pi_A$ is determined, and (16) determines $\Pi_C = T^{-1}(\Pi_A^\perp)$ and $C = S \cap \Pi_C$. $\square$

Deleting all exceptional-block edges would now give only an $O(q^9)$ bound on the expected loss, which does not guarantee that any fixed proportion of the $\Omega(q^9)$ copies remains. The smaller estimate $\mathbb{E}U(T) = O(q)$ allows mutual blocks to be removed outright. For the other blocks, we use their total incidence count to discard only edges with large load.

We now make the two preliminary deletions. First delete every edge lying in a mutual block, obtaining G_T^1 . For each $e \in E(G_T^1)$, define its *load*

$$\lambda_e := |\{B : B \text{ is an indexed nonmutual block and } e \in E(B)\}|.$$

All blocks in this definition retain their original vertex sets and edge sets from G_T^0 ; deletion does not redefine them. For a positive integer L , delete every edge with $\lambda_e > L$, obtaining G_T^{pre} . Since

$$\sum_{e \in E(G_T^1)} \lambda_e \leq Z(T),$$

at most $Z(T)/L$ edges are deleted in this second step. Every edge of G_T^{pre} lies in at most L indexed nonmutual blocks.

Let $K_{\text{pre}}(T)$ count designated copies whose edges all survive in G_T^{pre} . For every T , the number of removed edges is at most $U(T) + Z(T)/L$. Charging each destroyed copy to one of its deleted edges gives

$$K_{\text{pre}}(T) \geq K_0(T) - M_q \left(U(T) + \frac{Z(T)}{L} \right). \quad (24)$$

Taking expectations in this pointwise inequality and using (18), (22), and (23), we obtain absolute constants $c_1, C_1, C_2 > 0$ such that

$$\mathbb{E}K_{\text{pre}}(T) \geq c_1 q^9 - C_1 q^5 - \frac{C_2 q^9}{L}$$

for all sufficiently large q . Fix an integer $L \geq 4C_2/c_1$, and then take q sufficiently large that $C_1 q^5 \leq c_1 q^9/4$. Some T consequently satisfies

$$K_{\text{pre}}(T) \geq c_2 q^9, \quad c_2 := c_1/2 > 0. \quad (25)$$

Fix this transformation T for the remainder of the proof. The pointwise comparison (24) is what permits the copy count and both deletion costs to be handled together.

4 Pruning the blocks while preserving copies

Fix the transformation from (25). It remains to remove forbidden common neighborhoods inside the nonmutual blocks. Each block has at most four vertices on its controlled side, and we will retain at most two of them as neighbors of each vertex on its exceptional side. Such a restriction could destroy a designated copy with probability one if that copy needed three controlled-side vertices. The following geometric fact rules out this obstruction.

Lemma 8 (Intersection with controlled-side planes). *For every designated pair (A, C) and every $E \in \mathcal{D}$,*

$$|A \cap T(E)^\perp| \leq 2, \quad |C \cap T^{-1}(E^\perp)| \leq 2.$$

Proof. If Π_A and $T(E)^\perp$ share three points of A , then those points are noncollinear by Lemma 3, and $\Pi_A = T(E)^\perp$. Together with $\Pi_A = T(\Pi_C)^\perp$, this implies $E = \Pi_C$, contrary to $E \in \mathcal{D}$ and $\Pi_C \notin \mathcal{D}$.

Similarly, if Π_C and $T^{-1}(E^\perp)$ share three points of C , they are the same plane. Thus $T(\Pi_C) = E^\perp$, and $\Pi_A = T(\Pi_C)^\perp = E$, again a contradiction. $\square$

We can therefore choose two-subsets without making the survival of any designated copy impossible. For an indexed nonmutual block B , let X_B and Y_B denote its original labeled controlled and exceptional sides. Whenever $3 \leq |X_B| \leq 4$, choose, for every $y \in Y_B$, a uniformly random set

$$R_{B,y} \in \binom{X_B}{2}.$$

All choices are independent over indexed pairs (B, y) . Retain an edge e of G_T^{pre} precisely when, for every block B containing it with $|X_B| \geq 3$, writing $e = xy$ with $x \in X_B$ and $y \in Y_B$ gives $x \in R_{B,y}$. Blocks with $|X_B| \leq 2$ impose no restriction. The selections use the original sets X_B , even when some incident edges were deleted earlier. Let G be the resulting random subgraph.

Proposition 9 (Exclusion of the forbidden graph). *Every realization of the local pruning is $K_{3,5}$ -free.*

Proof. Take three vertices $p_1, p_2, p_3 \in S_L$. Their underlying points span a plane Π by the cap property. Their common right neighbors in G_T^0 correspond exactly to

$$S \cap E, \quad E := T^{-1}(\Pi^\perp).$$

If $E \notin \mathcal{D}$, there are at most four such neighbors by Proposition 4. If $E \in \mathcal{D}$, then $\Pi = T(E)^\perp$, and the triple belongs to the controlled side of B_E^I . A mutual block has no edges after the preliminary deletion. Otherwise, its controlled side has size three or four, and local pruning leaves every vertex on its exceptional side with at most two neighbors in that controlled side. Thus in this case the triple has no common right neighbor in G .

For a triple $r_1, r_2, r_3 \in S_R$, let Q be its spanning plane. Its common left neighbors in G_T^0 correspond exactly to

$$S \cap E, \quad E := T(Q)^\perp.$$

If $E \notin \mathcal{D}$, there are at most four. Otherwise, $Q = T^{-1}(E^\perp)$, so the triple belongs to the controlled side of B_E^{II} . If this block is mutual, its edges have all been deleted. If it is nonmutual, local pruning again leaves each exceptional-side vertex with at most two neighbors in the controlled side. The triple then has no common left neighbor.

No triple in either bipartition class has five common neighbors. Since the connected graph $K_{3,5}$ has a unique bipartition up to exchange of its parts, this excludes every copy of $K_{3,5}$ in G . $\square$

The pruning is thus effective for every outcome of the random choices. Its effect on designated copies is different: they require at most two neighbors from any one choice, and the load bound limits the number of choices that affect them. The following estimate quantifies both facts.

Proposition 10 (Survival of designated copies). *Some realization of the local pruning retains at least*

$$6^{-16L} c_2 q^9$$

designated copies.

Proof. Fix a designated copy D in G_T^{pre} . A local choice $R_{B,y}$ affects D only if D has an edge of B incident with y . The vertices of X_B that this choice must contain form the set

$$Q_{B,y} := \{x \in X_B : xy \in E(D)\}.$$

By Lemma 8, $d := |Q_{B,y}| \leq 2$. For $k := |X_B| \in \{3, 4\}$,

$$\Pr(Q_{B,y} \subseteq R_{B,y}) = \frac{\binom{k-d}{2-d}}{\binom{k}{2}} \geq \frac{1}{6}.$$

For a fixed block, distinct affecting choices correspond to disjoint sets of its edges in D , since they have different exceptional-side endpoints. The number of affecting choices is consequently at most the number of edge–block incidences of D , namely

$$\sum_{e \in E(D)} \lambda_e \leq 16L.$$

Independence of the choices therefore gives $\Pr(D \subseteq G) \geq 6^{-16L}$. Summing over the at least $c_2 q^9$ designated copies in G_T^{pre} proves that their expected surviving number is at least $6^{-16L} c_2 q^9$. Some realization attains this expectation. $\square$

Proof of Theorem 1. For every sufficiently large odd prime power q , Propositions 9 and 10 give a bipartite $K_{3,5}$ -free graph on $2q^3$ vertices with at least κq^9 copies of $K_{4,4}$, where $\kappa := 6^{-16L}c_2 > 0$ is absolute.

For sufficiently large n , choose the largest power of three $q = 3^k$ satisfying $2q^3 \leq n$. Then

$$2q^3 \leq n < 2(3q)^3 = 54q^3, \quad q^9 > \frac{n^3}{54^3}.$$

Apply the construction at this q , and add isolated vertices to obtain an n -vertex graph. This proves the lower bound with $c = \kappa/54^3$. Lemma 2 gives the upper bound. $\square$

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