

Exact semidegree extraction from minimum outdegree

Abstract

For integers $n \geq 1$ and $0 \leq d \leq n - 1$, we determine the largest integer c such that every n -vertex digraph of minimum outdegree at least d contains a nonempty subdigraph in which every vertex has indegree and outdegree at least c . Digraphs have no loops or parallel arcs; opposite arcs are allowed. The exact guarantee is the least nonnegative integer satisfying $d(d+1) \leq c(2n-c-1)$, and it is best possible even when every vertex of the ambient digraph has outdegree exactly d . Consequently, for each fixed $0 < \alpha < 1$, the sharp guarantee under minimum outdegree at least αn , divided by n , converges to $1 - \sqrt{1 - \alpha^2}$. This answers the dense extraction question of Grzesik, Rödl, and Volec (Innov. Graph Theory 2025). The proof combines an exact count of the arcs controlled by mixed indegree/outdegree deletion with a matching construction. In the construction, the remaining outdegree demands reduce to an integral flow problem with nested neighborhoods, for which the total capacity condition is also sufficient.

Note. This paper was generated entirely by AI, including MiMo, using an automated research pipeline developed by Chenghua Liu and Hanyu Li.

1 Introduction

Every digraph of positive minimum outdegree contains a directed cycle. Thus a one-sided degree condition already forces a subdigraph with positive degrees in both directions. The corresponding quantitative question is less immediate: if every vertex has at least d outneighbors, how large a minimum indegree and minimum outdegree can be guaranteed on a common set of vertices? Deleting vertices of small indegree is a natural first step, but it also removes outneighbors of the vertices that remain. The problem is to control this loss using only the original outdegree condition.

We consider finite loopless digraphs with at most one arc in each ordered direction between distinct vertices. Both opposite arcs may be present; such a pair is called a *digon*. For a nonempty digraph G , write $d_G^+(v)$ and $d_G^-(v)$ for the outdegree and indegree of v , and define

$$\delta^+(G) = \min_{v \in V(G)} d_G^+(v), \quad \delta^-(G) = \min_{v \in V(G)} d_G^-(v), \quad \delta^{+-}(G) = \min\{\delta^+(G), \delta^-(G)\}.$$

The extraction parameter is

$$q(G) = \max_{\emptyset \neq S \subseteq V(G)} \delta^{+-}(G[S]). \quad (1.1)$$

Allowing arbitrary subdigraphs gives the same maximum, since completing a subdigraph to the induced subdigraph on its vertex set cannot decrease its degrees. A hypothesis beyond arc density is necessary: a transitive tournament has $\binom{n}{2}$ arcs and $q(G) = 0$. Degree balance permits extraction from arc density in Eulerian digraphs [5]; here no balance between the two directions is assumed.

Grzesik, Rödl, and Volec [4, Theorem 1.1] proved that every n -vertex digraph with $\delta^+(G) \geq d$ satisfies

$$q(G) \geq \frac{d(d+1)}{2n}. \quad (1.2)$$

Their tournament constructions establish the sharp leading constant in the small-density regime. For d proportional to n , they gave additional estimates and asked for the optimal normalized bound

[4, Problem 1]. We determine the finite extremum, including the correction that becomes significant at positive density.

For integers $n \geq 1$ and $0 \leq d \leq n - 1$, put

$$\mathbf{q}(n, d) = \min_{\substack{|V(G)|=n \\ \delta^+(G) \geq d}} q(G) \quad (1.3)$$

and define

$$C(n, d) = \min \left\{ c \in \mathbb{Z}_{\geq 0} : \frac{d(d+1)}{2} \leq cn - \frac{c(c+1)}{2} \right\}. \quad (1.4)$$

The integer $c = d$ satisfies this inequality, so $0 \leq C(n, d) \leq d$.

Theorem 1.1. *For all integers $n \geq 1$ and $0 \leq d \leq n - 1$,*

$$\mathbf{q}(n, d) = C(n, d) = \left\lceil \frac{2n - 1 - \sqrt{(2n - 1)^2 - 4d(d+1)}}{2} \right\rceil. \quad (1.5)$$

An extremal digraph can be chosen with every vertex of outdegree exactly d .

For the dense problem, fix $0 < \alpha < 1$, set $N_\alpha = \lceil (1 - \alpha)^{-1} \rceil$, and for $n \geq N_\alpha$ define

$$h(n, \alpha) = \frac{\mathbf{q}(n, \lceil \alpha n \rceil)}{n}, \quad h(\alpha) = \liminf_{n \rightarrow \infty} h(n, \alpha). \quad (1.6)$$

The restriction on n is precisely the feasibility condition $\lceil \alpha n \rceil \leq n - 1$. Integrality of the degrees and the induced-subdigraph reduction show that this is the function in [4, Section 4.1].

Corollary 1.2. *For every fixed $0 < \alpha < 1$ and every integer $n \geq N_\alpha$,*

$$h(n, \alpha) = \frac{C(n, \lceil \alpha n \rceil)}{n}.$$

Moreover, the limit exists and

$$\lim_{n \rightarrow \infty} h(n, \alpha) = h(\alpha) = 1 - \sqrt{1 - \alpha^2}. \quad (1.7)$$

The previous dense estimates were

$$\max \left\{ \frac{\alpha^2}{2}, 1 - \sqrt{3 - 4\alpha + \alpha^2} \right\} \leq h(\alpha) \leq \alpha \quad [4, \text{Section 4.1}].$$

For example, at minimum outdegree $4n/5$, these estimates guarantee asymptotically $(1 - \sqrt{11}/5)n \approx 0.337n$, whereas the sharp leading term is $2n/5$. The difference comes from retaining the quadratic term $c(c+1)/2$ in (1.4). Omitting it recovers (1.2); retaining it changes the leading answer when c is linear in n .

The lower bound uses the mixed deletion procedure of [4, Section 2]: if no subdigraph has minimum semidegree greater than c , repeatedly delete a vertex with indegree or outdegree at most c . The difficulty is that the controlled direction may change from one deletion to the next. We mark the direction controlled on each vertex pair. The unmarked possible directions nevertheless form a transitive tournament, with outdegrees $0, 1, \dots, n - 1$. Whatever arcs of the original digraph occur in these directions, the marked directions must supply at least $d + (d - 1) + \dots + 1$ further outgoing arcs. Counting their capacity exactly gives (1.4).

Sharpness requires a compatible placement of those arcs. In an ordered construction, backward arcs supply most of the outdegree, leaving forward demands $d, d-1, \dots, 1$. Capping the forward indegree at c ensures that the largest vertex of every induced subdigraph has indegree at most c . The total capacity condition is necessary, but a priori smaller sets of demands could encounter tighter bottlenecks. We show that the nested allowed neighborhoods make every such cut no more restrictive than (1.4). An integral flow then places the forward arcs, and backward arcs complete every outdegree to d . Sections 2 and 3 prove these two assertions; Section 4 gives the dense limit and its relation to directed degeneracy.

2 The sharp peeling bound

The deletion order and the ambient outdegrees give opposite bounds on one set of arcs. We distinguish possible directions from arcs actually present in the digraph. For a real number x , write $x_+ = \max\{x, 0\}$.

Proposition 2.1. *Let G be an n -vertex digraph with $\delta^+(G) \geq d$, where $n \geq 1$ and $0 \leq d \leq n-1$ are integers. If $c = q(G)$, then*

$$\frac{d(d+1)}{2} \leq cn - \frac{c(c+1)}{2}. \quad (2.1)$$

Proof. Every nonempty induced subdigraph of G has minimum semidegree at most c . Repeatedly delete a vertex whose current outdegree or indegree is at most c , recording its type as O or I, respectively. If both conditions hold, choose either type. Label the vertices in reverse deletion order as $v_1, \dots, v_n$, so that the vertices present at each deletion form an initial segment. Thus v_i is deleted from $G[\{v_1, \dots, v_i\}]$.

For each pair $j < i$, mark one of its two possible arc directions by the rule

$$\begin{cases} v_i \rightarrow v_j, & \text{if } v_i \text{ has type O,} \\ v_j \rightarrow v_i, & \text{if } v_i \text{ has type I.} \end{cases} \quad (2.2)$$

The marking is defined whether or not the corresponding arc belongs to G . Let M be the set of arcs of G whose directions are marked. Charge each such arc to the endpoint of larger index. At most $\min\{c, i-1\}$ arcs are charged to v_i : the deletion bounds their number by c , and there are only $i-1$ earlier vertices. Every arc of M is charged exactly once. Since $0 \leq c \leq n-1$, this gives

$$|M| \leq \sum_{i=1}^n \min\{c, i-1\} = cn - \frac{c(c+1)}{2}. \quad (2.3)$$

To obtain the lower bound on $|M|$, form a tournament T from the unmarked possible direction on each vertex pair. It need not be a subdigraph of G , but it is transitive. To see this, build an order by inserting $v_1, v_2, \dots, v_n$ in turn. If v_i has type I, its unmarked directions point from v_i to every previously inserted vertex, so place it first. If it has type O, all those directions point toward v_i , so place it last. Inductively, every arc of T points forward in the resulting order. Its outdegrees are therefore $n-1, n-2, \dots, 0$.

For each vertex v , put $f(v) = d_T^+(v)$. At most $f(v)$ outgoing arcs of G at v can be unmarked. Since $d_G^+(v) \geq d$, at least $(d - f(v))_+$ are marked. Summing at the tails of the marked arcs yields

$$|M| \geq \sum_{v \in V(G)} (d - f(v))_+ = \sum_{r=0}^{n-1} (d - r)_+ = \frac{d(d+1)}{2}. \quad (2.4)$$

If a pair of vertices supports a digon, its two arcs lie in different classes, so this count neither omits nor counts an arc twice. Combining (2.3) and (2.4) proves (2.1). $\square$

Since $q(G)$ is an integer, Proposition 2.1 and the definition of $C(n, d)$ imply

$$q(n, d) \geq C(n, d). \quad (2.5)$$

3 An extremal construction

We now realize the two counts in the lower bound simultaneously. Fix integers $n \geq 1$ and $0 \leq d \leq n-1$, and set $c = C(n, d)$. For $d = 0$, take $G_{n,0}$ to be the edgeless digraph, so $q(G_{n,0}) = 0 = c$. Assume henceforth that $d \geq 1$, so $1 \leq c \leq d$.

Order the vertices as $[n] = \{1, \dots, n\}$. An arc $i \rightarrow j$ is *forward* if $i < j$, and *backward* if $i > j$. A vertex $i \leq d$ can use its $i - 1$ backward directions but still needs $d - i + 1$ forward arcs to reach outdegree d . Later vertices can reach outdegree d using backward arcs alone. We choose the required forward arcs so that no vertex receives more than c of them. This will bound the indegree of the largest vertex in every induced subdigraph. The only remaining issue is whether the demands can be met with these capacities at all vertices, not just in total.

Lemma 3.1. *There is a set $F \subseteq \{i \rightarrow j : 1 \leq i \leq d, i < j \leq n\}$ such that*

$$d_F^+(i) = d - i + 1 \quad (1 \leq i \leq d), \quad (3.1)$$

$$d_F^-(j) \leq c \quad (1 \leq j \leq n). \quad (3.2)$$

Here degrees in F are taken in the digraph with vertex set $[n]$ and arc set F .

Proof. Consider a network with source σ , sink τ , left vertices $L_1, \dots, L_d$, and right vertices $R_2, \dots, R_n$. Give the edge σL_i capacity $r_i = d - i + 1$, the edge $L_i R_j$ capacity one whenever $i < j$, and the edge $R_j \tau$ capacity c . We seek an integral flow of value

$$R_0 = \sum_{i=1}^d r_i = \frac{d(d+1)}{2},$$

which necessarily saturates every edge leaving the source.

Fix a set $S \subseteq [d]$ of indices of the left vertices on the source side of a cut. The source edges entering the other left vertices contribute $R_0 - \sum_{i \in S} r_i$. A right vertex R_j contributes c if placed on the source side, and $\deg_S(j) = |\{i \in S : i < j\}|$ if placed on the sink side. These choices are independent for different right vertices. Thus the minimum cut capacity with this left set is

$$R_0 - \sum_{i \in S} r_i + \sum_{j=2}^n \min\{c, \deg_S(j)\}.$$

The max-flow/min-cut criterion therefore reduces to

$$\sum_{i \in S} (d - i + 1) \leq \sum_{j=2}^n \min\{c, |\{i \in S : i < j\}|\} \quad \text{for every } S \subseteq [d]. \quad (3.3)$$

The allowed neighborhoods are nested. This lets us evaluate the cut capacity by the ordered positions of the selected left vertices. For $S = \emptyset$, both sides vanish. Otherwise write $S = \{i_1 < \dots < i_s\}$. For each $t \leq s$, the inequality $\deg_S(j) \geq t$ holds exactly when $j > i_t$. Counting these levels gives

$$\sum_{j=2}^n \min\{c, \deg_S(j)\} = \sum_{t=1}^{\min\{c, s\}} (n - i_t). \quad (3.4)$$

If $s \leq c$, the right side of (3.3) minus the left side is

$$\sum_{t=1}^s ((n - i_t) - (d - i_t + 1)) = s(n - d - 1) \geq 0.$$

For $s > c$, the first c selected positions cancel between the two sides. The left side of (3.3) minus its right side is

$$s(d + 1) - cn - \sum_{t=c+1}^s i_t \leq s(d + 1) - \frac{s(s + 1)}{2} + \frac{c(c + 1)}{2} - cn,$$

where we used $i_t \geq t$. Denote the final expression by $g(s)$. For $s < d$, one has $g(s + 1) - g(s) = d - s \geq 0$. Since $s \leq d$,

$$g(s) \leq g(d) = \frac{d(d + 1)}{2} + \frac{c(c + 1)}{2} - cn \leq 0$$

by (1.4). Thus the full set of demands is the only possible obstruction, and the threshold inequality rules it out. This proves (3.3) for every S .

Thus every source-sink cut has capacity at least R_0 . The max-flow/min-cut theorem gives a flow of value R_0 , and the integer capacities allow the flow to be chosen integral.

Select $i \rightarrow j$ precisely when the flow on $L_i R_j$ is one. The unit capacities make this a set of arcs. Saturation of the source edges gives (3.1), and the sink capacities give (3.2) for $j \geq 2$. For $j = 1$, the latter holds because no forward arc enters vertex 1. $\square$

Let $G_{n,d}$ start with the forward arcs supplied by Lemma 3.1. For each $i \leq d$, add all backward arcs $i \rightarrow j$ with $j < i$. For each $i > d$, add the arcs $i \rightarrow j$ for $1 \leq j \leq d$; these are backward because $d < i$. There are no other arcs. For $i \leq d$, the resulting outdegree is

$$d_{G_{n,d}}^+(i) = (d - i + 1) + (i - 1) = d,$$

and every $i > d$ has exactly the d added backward out-arcs. Thus all vertices have outdegree d .

For any nonempty $S \subseteq [n]$, let $j = \max S$. Every arc of $G_{n,d}[S]$ entering j is forward, hence belongs to F . It follows that

$$\delta^{+-}(G_{n,d}[S]) \leq d_{G_{n,d}[S]}^-(j) \leq d_F^-(j) \leq c.$$

Consequently $q(G_{n,d}) \leq C(n, d)$, proving the matching upper bound.

Completion of the proof of Theorem 1.1. The construction and (2.5) give $q(n, d) = C(n, d)$, including $d = 0$. They also show that $G_{n,d}$ is extremal and can have all outdegrees exactly d .

It remains to evaluate $C(n, d)$. Its defining inequality is equivalent to $p(c) \leq 0$, where

$$p(c) = c^2 - (2n - 1)c + d(d + 1).$$

Since $p(d) = 2d(d - n + 1) \leq 0$, the polynomial has real roots and d lies between them. Its smaller root is

$$\rho_{n,d} = \frac{2n - 1 - \sqrt{(2n - 1)^2 - 4d(d + 1)}}{2}.$$

The square root is at most $2n - 1$, so $0 \leq \rho_{n,d} \leq d$. The integer $\lceil \rho_{n,d} \rceil$ is at most d , and therefore still lies between the two roots. Every smaller nonnegative integer is below $\rho_{n,d}$, where p is positive. Hence $C(n, d) = \lceil \rho_{n,d} \rceil$, as claimed. In particular, $C(n, 0) = 0$ and $C(n, n - 1) = n - 1$, also when $n = 1$. $\square$

The network has $O(n)$ vertices and $O(dn)$ edges. Integral augmentation from zero flow increases its value by at least one per step, so at most $d(d+1)/2$ augmentations suffice. This gives a polynomial-time construction; c can be computed from the integer inequality (1.4) without approximating the square root.

4 The dense limit and directed degeneracy

Proof of Corollary 1.2. Fix $0 < \alpha < 1$ and put $d_n = \lceil \alpha n \rceil$. The finite identity follows from Theorem 1.1. With $\rho_n = \rho_{n,d_n}$, we have

$$h(n, \alpha) = \frac{\lceil \rho_n \rceil}{n}, \quad 0 \leq \frac{\lceil \rho_n \rceil - \rho_n}{n} < \frac{1}{n}.$$

Moreover,

$$\frac{\rho_n}{n} = \frac{1}{2} \left(2 - \frac{1}{n} - \sqrt{\left(2 - \frac{1}{n} \right)^2 - 4 \frac{d_n}{n} \frac{d_n + 1}{n}} \right) \longrightarrow 1 - \sqrt{1 - \alpha^2},$$

since $d_n/n \rightarrow \alpha$. The same limit therefore holds for $h(n, \alpha)$. $\square$

The value in (1.7) strictly improves both lower bounds for every $0 < \alpha < 1$. For the first, rationalization gives $1 - \sqrt{1 - \alpha^2} = \alpha^2 / (1 + \sqrt{1 - \alpha^2}) > \alpha^2 / 2$. For the second, the difference between the radicands is $(3 - 4\alpha + \alpha^2) - (1 - \alpha^2) = 2(1 - \alpha)^2 > 0$. The exact value is also less than α , because $1 - \alpha^2 > (1 - \alpha)^2$.

At the two endpoints,

$$\begin{aligned} h(\alpha) &= \frac{\alpha^2}{2} + \frac{\alpha^4}{8} + O(\alpha^6) & (\alpha \downarrow 0), \\ h(\alpha) &= 1 - \sqrt{2(1 - \alpha)} + O((1 - \alpha)^{3/2}) & (\alpha \uparrow 1). \end{aligned}$$

The first expansion recovers the sharp leading small-density term and identifies its next correction.

The finite parameter also describes directed degeneracy. Under the convention of [1, 3], a digraph is *weakly k -degenerate* if every nonempty subdigraph has a vertex of indegree or outdegree less than k ; hence $q(G) \leq c$ is equivalent to weak $(c + 1)$ -degeneracy. The “at most k ” convention in [2] shifts the index by one.

For oriented graphs, which forbid digons, Proposition 2.1 still applies, but the construction in Section 3 need not be oriented. Thus it does not settle [4, Problem 2]. Normalizing the extracted semidegree by its own order $|V(H)|$, instead of n , also gives a different problem [4, Section 4.3].

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