# The space of Delzant polytopes is an absolute retract

**Abstract**

For every $n \geq 1$, we prove that the space of full-dimensional Delzant polytopes in $\mathbb{R}^n$, with the Hausdorff or equivalently the symmetric-difference-volume topology, is an absolute extensor for metrizable spaces. It is therefore an absolute retract and admits a strong deformation retraction onto any prescribed polytope. Its open Hausdorff balls have the same extension property. This answers the fundamental-group question of Pelayo and Santos (Adv. Math., 2025), without restrictions on the normal fans occurring in a continuous family. The proof adapts the extension method of Dugundji (Pacific J. Math., 1951): smooth projective common refinements replace unrestricted convex interpolation, and barycentric subdivision organizes the resulting labels into refinement chains. A boundary estimate independent of the number of labels gives continuity at arbitrary prescribed closed subsets, even when infinitely many normal fans accumulate there. Away from these subsets, the extensions use only finitely many normal fans locally. We also construct continuous Delzant approximations of families of full-dimensional convex bodies with any prescribed positive continuous error, proving that the Delzant locus is homotopy dense in the space of such bodies. The extension theorem applies to the corresponding metric moduli spaces of symplectic toric manifolds with fixed torus and specified moment maps.

**Note.** This paper was generated entirely by AI, including MiMo, using an automated research pipeline developed by Chenghua Liu and Hanyu Li.

## 1 Introduction

Delzant's classification [3] identifies symplectic toric manifolds with convex polytopes whose local lattice geometry encodes smoothness. The metric topology of this class of polytopes describes continuous variation of moment images, including changes in their combinatorial type. Such changes are not controlled by the Hausdorff distance alone: arbitrarily small perturbations can introduce new facets, and a continuous family can encounter infinitely many normal fans. Understanding the topology of each fixed-fan stratum is therefore not enough to understand the full moduli space.

A full-dimensional polytope in $\mathbb{R}^n$ is *Delzant* if it is simple, its edge directions are rational, and the primitive edge vectors at each vertex form a basis of $\mathbb{Z}^n$. Equivalently, its normal fan is smooth: every cone is generated by part of a lattice basis. The facet support constants are arbitrary real numbers; no integrality condition is imposed on the vertices. Write $\mathcal{C}_p(n)$ for the space of nonempty compact convex subsets of $\mathbb{R}^n$ with nonempty interior, and let $\mathcal{D}(n) \subset \mathcal{C}_p(n)$ be its Delzant subspace. We consider the Hausdorff metric $d_{\mathrm{H}}$ and the symmetric-difference metric

$$d_{\mathrm{V}}(K, L) = \mathrm{Vol}_n(K \triangle L).$$

They induce the same topology on $\mathcal{C}_p(n)$, as recalled in Proposition 2.5.

Pelayo, Pires, Ratiu, and Sabatini [7] introduced this metric approach to toric moduli spaces. In dimension four they established connectedness and completion results and showed that the space of Delzant polygons is neither complete nor locally compact. Pelayo and Santos [8] subsequently proved path connectedness of $\mathcal{D}(n)$ in every dimension by passing to smooth common refinements of normal fans. They also proved that loops with finitely many normal fans, and more generally locally refining loops, are null-homotopic [8, Corollary 3.37 and Theorem 3.39]. Here a loop is locally refining if the fans of its nearby values refine the fan at each given parameter value. Every loop in $\mathcal{D}(2)$ has this property, but their three-dimensional Example 3.41 shows that it need not hold in higher dimension. Their Question 3.46 asks whether $\pi_1(\mathcal{D}(n))$ is nevertheless trivial. The weak contractibility proved in [8, Corollary 3.51] uses a strictly finer CW topology and does not settle this question for arbitrary metric-continuous loops.

We prove an extension theorem in the metric topology that is stronger than contractibility: every continuous map from a closed subset of a metrizable space into $\mathcal{D}(n)$ extends to the whole space. No dimension bound on the source, and no restriction on the normal fans in the prescribed image, is required.

**Definition 1.1.** A space $Y$ is an *absolute extensor for metrizable spaces* if, for every metric space $X$, every closed subset $A \subset X$, and every continuous map $f : A \to Y$, there is a continuous extension $F : X \to Y$. A metrizable space $Y$ is an *absolute retract* if every realization of $Y$ as a closed subspace of a metrizable space admits a retraction onto $Y$.

The extensor property implies the retract property by extending the inverse of a closed embedding. Neither definition imposes completeness or local compactness. To formulate the local assertion, identify a convex body $K$ with its support function

$$h_K(u) = \max_{x\in K}\langle u, x\rangle, \qquad u \in S^{n-1},$$

in the normed space $E_n = C(S^{n-1}, \mathbb{R})$ with the supremum norm. This identification is an isometric embedding for $d_{\mathrm{H}}$.

**Theorem 1.2** (Main theorem)**.** *For every integer $n \geq 1$, the space $\mathcal{D}(n)$, with either $d_{\mathrm{H}}$ or $d_{\mathrm{V}}$, is an absolute extensor for metrizable spaces. An extension $F : X \to \mathcal{D}(n)$ of a continuous map $f : A \to \mathcal{D}(n)$ from a closed subset of a metric space can be chosen so that every point of $X \setminus A$ has a neighborhood on which only finitely many normal fans occur.*

*The same assertions hold for every nonempty intersection $\mathcal{D}(n) \cap W$, where $W \subset E_n$ is open and convex. In particular, every nonempty open Hausdorff ball in $\mathcal{D}(n)$ is an absolute retract.*

**Corollary 1.3.** *For every $n \geq 1$ and every $P_* \in \mathcal{D}(n)$, there is a strong deformation retraction of $\mathcal{D}(n)$ onto $\{P_*\}$. Consequently, $\mathcal{D}(n)$ is path connected and locally contractible, and*

$$\pi_k(\mathcal{D}(n), P_*) = 0 \qquad (k \geq 1).$$

*In particular, Question 3.46 of [8] has an affirmative answer in every dimension.*

*Deduction from Theorem 1.2.* In the metric space $X = \mathcal{D}(n) \times [0, 1]$, consider the closed subset

$$A = (\mathcal{D}(n) \times \{0, 1\}) \cup (\{P_*\} \times [0, 1]).$$

Prescribe $f(P, 0) = P$, $f(P, 1) = P_*$, and $f(P_*, t) = P_*$. These prescriptions agree on overlaps and define a continuous map on $A$ by the pasting lemma for a finite closed cover. An extension is the required strong deformation retraction. Applying this argument to each open Hausdorff ball proves local contractibility. The remaining assertions follow from contractibility. □

The geometric obstruction to direct interpolation is that $\mathcal{D}(n)$ is not convex under Minkowski addition. The sum of two Delzant polytopes can have a singular normal fan, already in dimension two (Example 2.4). Interpolation does remain Delzant when the normal fans form a refinement chain: a convex combination has the finest fan among its positive summands. Smooth projective common refinements also admit metric control. If $S$ realizes such a refinement of the fans of $P, P_1, \dots, P_m$, then $P + tS$ has the fan of $S$ for every $t > 0$ and converges to $P$ as $t \downarrow 0$. These are established geometric ingredients; the use of small Minkowski perturbations in refinement arguments already appears in [8, §3.2.1 and Remark 3.42].

Our contribution is to assemble these finite choices into compatible extensions over arbitrary metric spaces. Dugundji's extension construction [4, Theorem 4.1] interpolates values in a convex target using a partition of unity. In our setting, labels at unrelated vertices cannot be interpolated directly. Instead, we assign a label to each face of the nerve of a locally finite cover. Its fan refines those of all proper-face labels, while its support function remains close to a prescribed target value. Barycentric subdivision turns nested faces into refinement chains, on which affine interpolation is allowed. Each choice has only finitely many constraints, even when the nerve has unbounded dimension.

The remaining issue is continuity at the prescribed closed subset. There the cover is chosen at a scale comparable to the distance from that subset. As a point approaches a boundary value, all labels contributing to its image approach that same value. The error of a Minkowski convex combination is at most the largest error of its summands, independently of their number. This gives the uniform boundary control needed for arbitrary accumulations of normal fans. We formulate the argument as an extension criterion for preordered subsets of normed spaces in Theorem 3.1. Applied to families of convex bodies, it also yields continuous Delzant approximation with prescribed pointwise error and the homotopy-density theorem (Theorem 4.4). A quantitative disk filling is given in Appendix A.

## 2 Convex bodies and smooth refinements

Let $B$ denote the closed Euclidean unit ball in $\mathbb{R}^n$. Support functions determine compact convex sets, and

$$d_{\mathrm{H}}(K, L) = \|h_K - h_L\|_\infty, \qquad h_{\alpha K + \beta L} = \alpha h_K + \beta h_L \quad (\alpha, \beta \geq 0). \tag{2.1}$$

Here addition is Minkowski addition, and a summand with coefficient zero is $\{0\}$. The first identity follows from

$$K \subset L + \delta B \iff h_K \leq h_L + \delta,$$

using separation of closed convex sets. The second follows by maximizing independently in the two summands.

The subset $\mathcal{C}_p(n) \subset E_n$ is convex. A convex combination has a positive coefficient, and the corresponding dilated body contains an open ball whose translate lies in the combination. We will repeatedly use the estimate

$$d_{\mathrm{H}}\left(\sum_{j=0}^{k} \lambda_j P_j, K\right) \leq \sum_{j=0}^{k} \lambda_j d_{\mathrm{H}}(P_j, K) \leq \max_j d_{\mathrm{H}}(P_j, K), \qquad \lambda_j \geq 0, \quad \sum_j \lambda_j = 1. \tag{2.2}$$

In particular, Hausdorff balls in $\mathcal{C}_p(n)$ are convex in support-function coordinates.

For a nonempty face $F$ of a full-dimensional polytope $P$, its normal cone consists of the vectors whose linear functionals are maximized on all of $F$. These cones form the complete pointed normal fan $\mathcal{N}(P)$. A fan $\Sigma'$ *refines* $\Sigma$ if each cone of $\Sigma'$ is contained in a cone of $\Sigma$. A fan is *rational* if its cones are generated by lattice vectors. It is *polytopal*, or *projective*, if it is the normal fan of a polytope.

A rational simplicial cone is *smooth* if its primitive ray generators extend to a basis of $\mathbb{Z}^n$. In full dimension this means that their determinant has absolute value one. A fan is smooth if all its cones are smooth; for a complete fan, it suffices to check its maximal cones.

For positive coefficients, the normal fan of a Minkowski sum is the common refinement of the normal fans of its summands:

$$\mathcal{N}(\alpha P + \beta Q) = \mathcal{N}(P) \wedge \mathcal{N}(Q) \qquad (\alpha, \beta > 0). \tag{2.3}$$

The common refinement on the right consists of the intersections of cones in the two fans. To obtain the identity, observe that the face exposed by a vector in the sum is the sum of the exposed faces in the summands. Equivalently, the domains of linearity of the sum of two convex support functions are the common refinement of their domains of linearity.

**Lemma 2.1** (Interpolation along a chain)**.** *Suppose $P_0, \dots, P_k \in \mathcal{D}(n)$ and $\mathcal{N}(P_j)$ refines $\mathcal{N}(P_i)$ whenever $i < j$. Every convex combination $P = \sum_{i=0}^{k} \lambda_i P_i$ is Delzant. More precisely, if $j = \max\{i : \lambda_i > 0\}$, then $\mathcal{N}(P) = \mathcal{N}(P_j)$.*

*Proof.* Discard the zero coefficients. Positive dilation does not change a normal fan, and the common refinement of a finite chain of fans is its finest member. Apply (2.3) repeatedly. ☐

We next recall the resolution argument that supplies smooth projective common refinements. This is the toric resolution-of-singularities construction [2, §11.1]; the polytopal statements are recorded in [8, Proposition 2.13 and Theorem 2.15]. We include the argument to make the geometric input explicit.

**Lemma 2.2** (Smooth projective refinement)**.** *Every rational polytopal complete fan has a smooth polytopal refinement. Consequently, the normal fans of finitely many full-dimensional polytopes with rational normal directions have a common smooth polytopal refinement.*

*Proof.* A stellar subdivision at a new ray replaces each cone containing the ray by the cones spanned by that ray and the faces not containing it, and includes all faces of the resulting cones. Such a subdivision of a polytopal fan remains polytopal. To see this, let $u$ be a nonzero vector on the new ray, let $P$ realize the fan, and let $F$ be the face on which $\langle u, \cdot \rangle$ is maximized. Choose $\epsilon > 0$ smaller than

$$h_P(u) - \langle u, v \rangle$$

for every vertex $v$ outside $F$. Cutting $P$ by

$$\langle u, x \rangle \le h_P(u) - \epsilon$$

retains those vertices and replaces the vertices in $F$ by new vertices on the edges leaving $F$. The normal cone at each new vertex is generated by $u$ and the normal cone of its original edge. These are precisely the maximal cones of the stellar subdivision. If the inserted ray is rational, the new fan is rational as well.

First make the fan simplicial. Choose a rational interior ray in every cone of dimension at least two, and perform the stellar subdivisions in decreasing order of dimension. The resulting barycentric subdivision is simplicial and remains polytopal.

For a maximal simplicial cone with primitive generators $v_1, \dots, v_n$, its multiplicity is

$$m = |\det(v_1, \dots, v_n)|\,.$$

The half-open fundamental parallelepiped represents the $m$ cosets of the sublattice generated by these vectors. If $m > 1$, it therefore contains a nonzero lattice point

$$w = \sum_{i=1}^{n} a_i v_i, \qquad 0 \le a_i < 1.$$

Let $u$ be the primitive vector on its ray. Writing $u = \sum_i b_i v_i$ gives $0 \le b_i < 1$. Let $\tau$ be the face generated by the $v_i$ with $b_i > 0$. This face has dimension at least two: a primitive lattice vector has no nonzero lattice multiple with coefficient strictly between zero and one.

Stellar subdivision at $u \in \operatorname{relint}(\tau)$ replaces each maximal cone containing $\tau$ by cones obtained by replacing one generator $v_i$ of $\tau$ by $u$. Multilinearity of the determinant shows that each new multiplicity is $b_i$ times the old multiplicity, hence is a strictly smaller positive integer. The same calculation holds for every affected maximal cone, since the coefficients of $u$ in the rays of their common face $\tau$ are unchanged.

At each step choose a cone of largest multiplicity $M > 1$. The number of cones of multiplicity $M$ strictly decreases, and no cone of multiplicity at least $M$ is introduced. After finitely many steps the largest multiplicity decreases. Induction on $M$ terminates with a smooth fan. Every step preserves projectivity.

For finitely many polytopes, start with their Minkowski sum. By (2.3), its normal fan is rational, polytopal, and refines all the given fans. Apply the preceding construction. $\square$

**Lemma 2.3** (Common refinements close to a target)**.** *Let $P, P_1, \ldots, P_m \in \mathcal{D}(n)$ and $\eta > 0$. There exists $Q \in \mathcal{D}(n)$ such that*

$$d_{\mathrm{H}}(Q, P) < \eta, \qquad \mathcal{N}(Q) \text{ refines } \mathcal{N}(P), \mathcal{N}(P_1), \ldots, \mathcal{N}(P_m).$$

*If $P$ belongs to an open subset $W \subset E_n$, then $Q$ can also be required to belong to $W$.*

*Proof.* Choose a polytope $S$ whose smooth projective fan refines all the listed fans, using Lemma 2.2. For $t > 0$, the polytope $Q_t = P + tS$ has normal fan $\mathcal{N}(S)$ by (2.3), and

$$d_{\mathrm{H}}(Q_t, P) = t\,\|h_S\|_\infty \longrightarrow 0.$$

Thus $Q_t$ is Delzant, and sufficiently small $t$ gives the desired error and, when required, membership in $W$. $\square$

The approximation error can be prescribed independently of the number of cones in the common refinement. In the extension argument, this lets the fan complexity grow near the prescribed data while the metric error tends to zero.

**Example 2.4** (Minkowski sums need not be Delzant)**.** Let

$$A = \operatorname{conv}\{(0,0), (1,0), (0,1)\}, \qquad B' = \operatorname{conv}\{(0,0), (-1,0), (0,1)\}.$$

Both polygons are Delzant. The primitive outward rays of the normal fan of $A + B'$, in counter-clockwise order, are

$$(1,0),\ (1,1),\ (-1,1),\ (-1,0),\ (0,-1).$$

The adjacent pair $(1,1), (-1,1)$ has determinant 2, so $A + B'$ is not Delzant. Taking products gives Delzant three-dimensional polytopes $P = A \times [0,1]$ and $Q = B' \times [0,1]$ whose sum is not Delzant.

Insert the planar ray $(0, 1)$. All six consecutive determinants are then one, and the resulting smooth fan is realized by the hexagon

$$T = \operatorname{conv}\{(2, -2), (2, 1), (1, 2), (-1, 2), (-2, 1), (-2, -2)\}.$$

Equivalently, $T$ is given by

$$-2 \le x \le 2, \qquad -2 \le y \le 2, \qquad x + y \le 3, \quad -x + y \le 3.$$

Thus $S = T \times [0, 1]$ has a smooth fan refining those of both $P$ and $Q$. For every $t > 0$, the polytope $R_t = P + tS$ is Delzant and its fan refines both given fans, whereas

$$d_{\mathrm{H}}(R_t, P) = t \left\|h_S\right\|_\infty = 3t.$$

Indeed, $\max_{x \in S} \|x\| = 3$, attained at $(2, -2, 1)$. This gives an explicit instance of Lemma 2.3.

## 2.1 Comparison of the metrics

The following equivalence connects the support-function description with the symmetric-difference topology used for toric moduli spaces; see [8, Proposition 3.3] and [9].

**Proposition 2.5.** *The Hausdorff metric and the volume-of-symmetric-difference metric induce the same topology on $\mathcal{C}_p(n)$.*

*Proof.* Fix $K \in \mathcal{C}_p(n)$ and translate so that $rB \subset K$ for some $r > 0$. If $\delta = d_{\mathrm{H}}(K, L) < r$, then $h_K - \delta \le h_L \le h_K + \delta$ and $h_K \ge r$. Hence

$$(1 - \delta/r)K \subset L \subset (1 + \delta/r)K,$$

and therefore

$$d_{\mathrm{V}}(K, L) \le \left((1 + \delta/r)^n - (1 - \delta/r)^n\right) \operatorname{Vol}_n(K).$$

This tends to zero as $\delta \to 0$.

Conversely, suppose $d_{\mathrm{V}}(K, L_j) \to 0$. Let $c > 0$ be the volume of a cap of $rB$ cut off by a hyperplane at distance $r/2$ from its center. If $(r/2)B$ were not contained in $L_j$, separation would give a unit vector $u$ with $h_{L_j}(u) < r/2$. The cap

$$rB \cap \{x : \langle u, x\rangle > r/2\}$$

would lie in $K \setminus L_j$, contradicting $d_{\mathrm{V}}(K, L_j) < c$. Thus $r_0 B \subset L_j$ eventually, where $r_0 = r/2$.

The volumes of these $L_j$ are uniformly bounded because $\operatorname{Vol}_n(L_j) \le \operatorname{Vol}_n(K) + d_{\mathrm{V}}(K, L_j)$. For any nonzero $x \in L_j$, the convex hull of $x$ and the radius-$r_0$ disk in the hyperplane $x^\perp$ through the origin lies in $L_j$. Its volume is

$$\frac{\kappa_{n-1} r_0^{n-1} \|x\|}{n},$$

where $\kappa_j$ denotes the volume of the unit $j$-ball and $\kappa_0 = 1$. Thus all sufficiently large $L_j$, and $K$, lie in a fixed ball $RB$.

For two bodies $K_1, K_2$ with $r_0 B \subset K_1, K_2 \subset RB$, put $\delta = d_{\mathrm{H}}(K_1, K_2)$. If $\delta > 0$, interchange them if necessary and choose a unit vector $u$ and $x \in K_1$ such that

$$\langle u, x\rangle = h_{K_1}(u) = h_{K_2}(u) + \delta.$$

Set $\lambda = \delta/(2(R + r_0))$. Since $0 < \delta \le 2R$, we have $0 < \lambda < 1$. By convexity, the ball

$$(1 - \lambda)x + \lambda r_0 B$$

is contained in $K_1$. Every point of this ball has $u$-coordinate at least

$$h_{K_1}(u) - \lambda(R + r_0) = h_{K_2}(u) + \delta/2,$$

so the ball is disjoint from $K_2$. Consequently,

$$d_{\mathrm{V}}(K_1, K_2) \ge \kappa_n \left( \frac{r_0\, d_{\mathrm{H}}(K_1, K_2)}{2(R + r_0)} \right)^n .$$

Apply this to $K$ and $L_j$ to obtain $d_{\mathrm{H}}(K, L_j) \to 0$. Convergence of sequences therefore agrees for the two metrics, which proves equality of their topologies. $\square$

The common interior ball is essential in this comparison. All bodies and all prescribed limiting values in this paper are full-dimensional.

## 3 Extension along refinement chains

We now isolate the topological argument. It replaces convexity of the whole target in Dugundji's construction [4] by convexity along chains in a preorder. A preorder is a reflexive, transitive relation; antisymmetry is not required.

**Theorem 3.1** (Extension by refinement chains)**.** *Let $Y$ be a nonempty subset of a real normed vector space $E$, with its subspace topology, and let $\preceq$ be a preorder on $Y$. Suppose the following conditions hold.*

(i) *For every finite collection $y_1, \dots, y_m \in Y$, every $y \in Y$, and every $\eta > 0$, there exists $z \in Y$ such that $y_i \preceq z$ for all $i$ and $\|z - y\| < \eta$.*

(ii) *For every finite chain $y_0 \preceq \cdots \preceq y_k$, its entire convex hull is contained in $Y$.*

*Then $Y$ is an absolute extensor for metrizable spaces.*

We first describe the interpolation used in the proof. Let $\{V_i\}_{i \in I}$ be a locally finite open cover of a metric space $Z$, and choose a locally finite partition of unity $\{\varphi_i\}_{i \in I}$ subordinate to it. Thus $\varphi_i \ge 0$, $\sum_i \varphi_i = 1$, and $\varphi_i(x) > 0$ implies $x \in V_i$. Such refinements and partitions exist by paracompactness of metric spaces [6].

The nerve $K$ has a simplex for every nonempty finite set $\sigma \subset I$ with $\bigcap_{i \in \sigma} V_i \ne \varnothing$. Its barycentric subdivision has a vertex

$$b_\sigma = \frac{1}{|\sigma|} \sum_{i \in \sigma} e_i$$

for each such simplex, where $e_i$ are the formal coordinate vertices. Its simplices are the flags $\sigma_0 \subsetneq \cdots \subsetneq \sigma_k$. Suppose that labels $q_\sigma \in Y$ have been chosen so that

$$q_\tau \preceq q_\sigma \qquad (\varnothing \ne \tau \subsetneq \sigma). \tag{3.1}$$

Send $b_\sigma$ to $q_\sigma$ and interpolate affinely on each subdivided simplex. The definitions agree on common faces, and condition (ii) puts every interpolated value in $Y$. Evaluate this interpolation at

$$p(x) = \sum_i \varphi_i(x) e_i$$

to obtain a map $G : Z \to Y$.

Only finite subcomplexes are needed to interpret this construction locally. Near any point of $Z$, a neighborhood meets only finitely many cover elements, so $p$ and the interpolation factor through the subdivision of a finite subcomplex. There the interpolation is an ordinary continuous piecewise affine map. It follows that $G$ is continuous, without a dimension bound or local-finiteness assumption on the nerve itself.

We also record which labels can contribute at a point. If $q_\sigma$ has positive coefficient in the interpolated value $G(x)$, then

$$\sigma \subset I(x) := \{i : \varphi_i(x) > 0\}. \tag{3.2}$$

Indeed, the coordinates of $b_\sigma$ are strictly positive on $\sigma$ and nonnegative elsewhere. A positive coefficient of $b_\sigma$ cannot contribute to a zero coordinate of $p(x)$. In particular, $x \in V_i$ for every $i \in \sigma$.

The interpolation has an explicit formula. List the positive partition weights at $x$ in decreasing order as $a_1 \geq \cdots \geq a_m > 0$, put $a_{m+1} = 0$, and let $I_j$ be the indices of the first $j$ weights. Then

$$G(x) = \sum_{j=1}^{m} j(a_j - a_{j+1}) q_{I_j}. \tag{3.3}$$

The coefficients are nonnegative and sum to $\sum_j a_j = 1$. Moreover, the coordinate with weight $a_\ell$ in the corresponding sum of barycenters equals $\sum_{j=\ell}^{m}(a_j - a_{j+1}) = a_\ell$. At a tie $a_j = a_{j+1}$, the potentially order-dependent term has coefficient zero. Thus the formula is independent of the ordering within ties and agrees across the subdivided simplices.

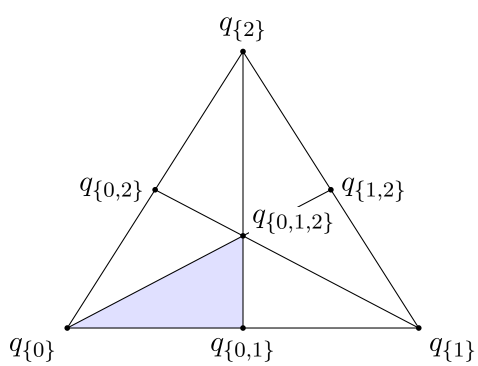


Figure 1: Labels on a subdivided triangle. On the shaded simplex they form the chain $q_{\{0\}} \preceq q_{\{0,1\}} \preceq q_{\{0,1,2\}}$. Unrelated labels at the original vertices are not directly interpolated.

*Proof of Theorem 3.1.* Let $A$ be closed in a metric space $(X, d)$, and let $f : A \to Y$ be continuous. If $A$ is empty, take a constant extension. If $A = X$, take $F = f$. Otherwise put $Z = X \setminus A$ and $r(z) = d(z, A) > 0$ for $z \in Z$.

Cover $Z$ by the open balls $B_d(z, r(z)/4)$ and take a locally finite open refinement $\{V_i\}_{i \in I}$. Choose $z_i \in Z$ so that, writing $r_i = d(z_i, A)$, we have

$$V_i \subset B_d(z_i, r_i/4). \tag{3.4}$$

Choose $a_i \in A$ satisfying

$$d(z_i, a_i) < 2r_i. \tag{3.5}$$

Such a point exists by the definition of the distance to $A$. Choose a subordinate partition of unity and let $K$ be the nerve of the cover.

For every nonempty simplex $\sigma$, choose an index $i(\sigma) \in \sigma$. Inductively over its dimension, choose $q_\sigma \in Y$ satisfying

$$\left\| q_\sigma - f(a_{i(\sigma)}) \right\| < r_{i(\sigma)}, \tag{3.6}$$

$$q_\tau \preceq q_\sigma \quad (\varnothing \neq \tau \subsetneq \sigma). \tag{3.7}$$

For a vertex, take the target itself as its label. For each higher-dimensional simplex, condition (i) applies to the finitely many proper-face labels. Thus the choices are possible in every dimension, even if $K$ has unbounded dimension.

The interpolation just described gives a continuous map $G : Z \to Y$. Define $F = G$ on $Z$ and $F = f$ on $A$. It remains to prove continuity at $A$.

If $x \in V_i$, the fact that distance to $A$ is 1-Lipschitz gives

$$d(x, A) > r_i - r_i/4 = 3r_i/4.$$

Together with (3.5), this implies

$$r_i < \frac{4}{3} d(x, A), \qquad d(x, a_i) < \frac{9}{4} r_i < 3d(x, A). \tag{3.8}$$

Fix $a \in A$ and $x \in Z$. Every label $q_\sigma$ contributing positively to $G(x)$ has $x \in V_{i(\sigma)}$, by (3.2). Therefore

$$\left\| q_\sigma - f(a_{i(\sigma)}) \right\| < \tfrac{4}{3} d(x, a), \tag{3.9}$$

$$d(a_{i(\sigma)}, a) < 4d(x, a). \tag{3.10}$$

By continuity of $f$ at $a$, the values $f(a_{i(\sigma)})$ are uniformly close to $f(a)$ over all contributing labels when $x$ is close to $a$. The other error in (3.9) also tends to zero. Since $G(x)$ is a convex combination of these labels, its error is bounded by the largest of their errors.

More explicitly, for $x \in Z$ sufficiently close to $a$,

$$\|F(x) - f(a)\| \leq \frac{4}{3} d(x, a) + \sup_{\substack{b \in A \\ d(b,a) < 4d(x,a)}} \|f(b) - f(a)\|. \tag{3.11}$$

The supremum is finite near $a$ and tends to zero as $x \to a$. Thus $F(x) \to f(a)$ from $Z$; continuity on $A$ handles approaches within $A$. This proves continuity of the extension on $X$. $\square$

The estimate (3.11) does not depend on the number of labels contributing at $x$. Local finiteness is needed only away from $A$, where it makes the interpolation finite and continuous.

# 4 Delzant extensions and approximation

*Proof of Theorem 1.2.* The space $\mathcal{D}(n)$ is nonempty, since it contains the unit cube. Using the support-function embedding, define a preorder on it by

$$P \preceq Q \iff \mathcal{N}(Q) \text{ refines } \mathcal{N}(P).$$

Distinct polytopes can have the same fan, so antisymmetry is neither asserted nor needed. Lemma 2.3 gives condition (i) of Theorem 3.1, and Lemma 2.1 gives condition (ii). The criterion proves the absolute extensor assertion for the Hausdorff topology.

On each subdivided simplex, the interpolated polytopes have only normal fans among those of its labels, by Lemma 2.1. Near each point of $X \setminus A$, the construction involves only a finite subcomplex and hence finitely many labels. Only finitely many normal fans therefore occur on that neighborhood. The constant extension when $A$ is empty has this property as well.

Now let $W \subset E_n$ be open and convex, with $Y = \mathcal{D}(n) \cap W \neq \varnothing$. By Lemma 2.3, a common upper bound can be chosen sufficiently close to a target in $Y$ to remain in $W$. Convex combinations along chains remain in $W$ by convexity. The same criterion and construction therefore apply to $Y$. An open Hausdorff ball is the intersection with an open norm ball in $E_n$, so it is a particular case.

Finally, Proposition 2.5 identifies the Hausdorff and symmetric-difference topologies. The extension and retract properties are topological, so the conclusions hold for both metrics. ☐

**Corollary 4.1** (Filling arbitrary spheres). *Every continuous map $f : S^k \to \mathcal{D}(n)$, for $k \geq 0$, extends to a continuous map $F : B^{k+1} \to \mathcal{D}(n)$, where $B^{k+1}$ is the closed unit ball. The extension can be chosen so that only finitely many normal fans occur on each compact subset of* $\operatorname{int} B^{k+1}$. *If $f(S^k)$ lies in an open Hausdorff ball, the filling can be chosen in that ball.*

*Proof.* Apply Theorem 1.2 with the sphere as the prescribed closed subset, using either $\mathcal{D}(n)$ or the indicated open Hausdorff ball as target. A compact subset of the interior is covered by finitely many neighborhoods on each of which only finitely many fans occur. ☐

## 4.1 Approximation of families of convex bodies

The same interpolation permits arbitrary convex bodies as target values. The density of Delzant polytopes in $\mathcal{C}_p(n)$ is already part of [8, Theorem 3.18]. We need approximation compatible with finitely many prescribed fans; the resulting parametric construction will then produce a continuous approximating family.

**Lemma 4.2.** *Given $K \in \mathcal{C}_p(n)$, finitely many polytopes $P_1, \dots, P_m \in \mathcal{D}(n)$, and $\eta > 0$, there exists $Q \in \mathcal{D}(n)$ with $d_{\mathrm{H}}(Q, K) < \eta$ whose fan refines every $\mathcal{N}(P_i)$. The collection of prescribed polytopes may be empty.*

*Proof.* Choose a finite $\eta/4$-net in $K$ and add $n+1$ affinely independent points of $K$. The convex hull of these points lies in $K$ and is at Hausdorff distance less than $\eta/4$ from $K$. Perturb the points to rational points by less than $\eta/4$, with a smaller perturbation if needed to preserve affine independence. The resulting polytope $R$ is full-dimensional and has rational vertices, and $d_{\mathrm{H}}(R, K) < \eta/2$.

The normal fan of $R + P_1 + \cdots + P_m$ is rational and polytopal. By Lemma 2.2, choose a polytope $S$ realizing a smooth projective refinement of this fan. Then $Q = R + tS$ has normal fan $\mathcal{N}(S)$ for every $t > 0$, and

$$d_{\mathrm{H}}(Q, R) = t \, \|h_S\|_\infty \, .$$

Choose $t$ so that this is less than $\eta/2$. The resulting $Q$ has the required approximation and refinement properties. ☐

**Proposition 4.3** (Controlled approximation). *Let $X$ be a metric space, $f : X \to \mathcal{C}_p(n)$ continuous, and $\epsilon : X \to (0, \infty)$ continuous. There exists a continuous map $g : X \to \mathcal{D}(n)$ such that*

$$d_{\mathrm{H}}(g(x), f(x)) < \epsilon(x) \qquad (x \in X).$$

*Every point of $X$ has a neighborhood on which the values of $g$ have only finitely many normal fans.*

*Proof.* For each $a \in X$, choose an open neighborhood $U_a$ such that

$$d_{\mathrm{H}}(f(x), f(a)) < \epsilon(a)/8, \qquad \epsilon(x) > \epsilon(a)/2 \quad (x \in U_a).$$

Take a locally finite open refinement $\{V_i\}$, with $V_i \subset U_{a_i}$, and a subordinate partition of unity. For each nonempty simplex $\sigma$ of the nerve, choose an index $i(\sigma) \in \sigma$. Inductively choose a label $Q_\sigma$ whose normal fan refines those of all proper-face labels and such that

$$d_{\mathrm{H}}(Q_\sigma, f(a_{i(\sigma)})) < \epsilon(a_{i(\sigma)})/8.$$

Lemma 4.2 supplies every choice, including vertex labels when there are no refinement constraints. Interpolate on the barycentric subdivision as in Section 3.

Every label contributing at $x$ satisfies $x \in V_{i(\sigma)}$, so its distance from $f(x)$ is less than

$$\epsilon(a_{i(\sigma)})/4 < \epsilon(x)/2.$$

The convex-combination estimate (2.2) gives the required error bound for $g(x)$. Locally, the interpolation involves only finitely many labels and is piecewise affine, proving continuity. Each interpolated value has the fan of one of those labels by Lemma 2.1, which proves the last assertion. □

A subset $Y$ of a space $Z$ is *homotopy dense* if there is a homotopy $H : Z \times [0,1] \to Z$ with $H(z,0) = z$ and $H(z,t) \in Y$ whenever $t > 0$. In the present setting, the homotopy can be chosen with a uniform metric bound.

**Theorem 4.4** (Homotopy density)**.** *There exists a continuous map*

$$H : \mathcal{C}_p(n) \times [0,1] \longrightarrow \mathcal{C}_p(n)$$

*such that* $H(K,0) = K$ *and, for every* $t > 0$*,*

$$H(K,t) \in \mathcal{D}(n), \qquad d_{\mathrm{H}}(H(K,t), K) < t.$$

*Proof.* Apply Proposition 4.3 on $X = \mathcal{C}_p(n) \times (0,1]$ to the projection $(K,t) \mapsto K$, with error function $\epsilon(K,t) = t$. Set $H(K,0) = K$. For any $K_0 \in \mathcal{C}_p(n)$ and $t > 0$,

$$d_{\mathrm{H}}(H(K,t), K_0) < t + d_{\mathrm{H}}(K, K_0),$$

which proves joint continuity at $(K_0, 0)$. Continuity elsewhere is part of the approximation construction. □

This also gives a direct contraction of $\mathcal{D}(n)$. Fix $P_* \in \mathcal{D}(n)$ and put

$$C(P,s) = \begin{cases} H\big((1-s)P + sP_*,\ s(1-s)\big), & 0 < s < 1, \\ P, & s = 0, \\ P_*, & s = 1. \end{cases}$$

For $0 < s < 1$ its values are Delzant. The approximation bound and continuity of Minkowski addition give continuity at both endpoints. This contraction need not fix $P_*$ at intermediate times; the strong contraction is supplied by Corollary 1.3.

## 5 Symplectic toric moduli

Fix a torus $T^n$, its integral lattice, and a lattice-compatible identification $\mathfrak{t}^* \cong \mathbb{R}^n$. A symplectic toric manifold here consists of a compact connected symplectic $2n$-manifold $(M, \omega)$, an effective Hamiltonian $T^n$-action, and a specified moment map $\mu : M \to \mathfrak{t}^*$. Isomorphisms are equivariant symplectomorphisms preserving the specified moment maps.

The Atiyah–Guillemin–Sternberg convexity theorem and Delzant's classification [1, 5, 3] imply that

$$[(M, \omega, T^n, \mu)] \longmapsto \mu(M)$$

is a bijection from the moduli set $\mathcal{M}(2n)$ of these isomorphism classes to $\mathcal{D}(n)$. The existence direction starts with facet inequalities $\langle x, u_i \rangle \geq \lambda_i$, where $u_i$ are primitive inward normals. The lattice map $\mathbb{Z}^m \to \mathbb{Z}^n$, $e_i \mapsto u_i$, is surjective because the normals at a vertex form a lattice basis. Its torus kernel is therefore a subtorus of $T^m$. Delzant's construction reduces $\mathbb{C}^m$ by this subtorus at the level determined by the $\lambda_i$. The basis condition gives freeness on the reduction level and hence smoothness; boundedness of the polytope gives compactness. The real support constants are symplectic parameters, explaining why the vertices need not be integral.

As in [7, 8], endow $\mathcal{M}(2n)$ with the pullback metric

$$d_{\mathcal{M}}([M], [M']) = \mathrm{Vol}_n\left(\mu(M) \triangle \mu'(M')\right).$$

The Delzant correspondence is then an isometry.

**Corollary 5.1.** *For every $n \geq 1$, the metric space $\mathcal{M}(2n)$ with fixed torus and specified moment maps is an absolute extensor for metrizable spaces and is an absolute retract. It strongly deformation retracts onto any specified isomorphism class. Every continuous map from a sphere into this moduli space extends to a ball, with only finitely many normal fans on each compact subset of its interior. In particular, $\pi_1(\mathcal{M}(6)) = 0$.*

*Proof.* Transport Theorem 1.2 and Corollaries 1.3 and 4.1 through the Delzant correspondence. □

For the associated toric varieties, refinement of normal fans gives a proper toric birational morphism [2]. The stellar subdivisions used above are toric modifications, and the perturbation $P + tS$ supplies a strictly convex support function on the common refinement. Thus the finite geometric step is a resolution of the fan of a Minkowski sum, followed by a choice of its support function.

The metric topology concerns isomorphism classes with the fixed torus and moment-map data. Along a continuous path, facets may appear or disappear and the underlying differentiable manifold may change; such a path is not, in general, a smooth family on a fixed manifold or a smooth fiber bundle. The construction is not claimed to be equivariant under changes of lattice coordinates. No conclusion about a further affine-unimodular quotient is asserted here.

## References


[1] M. F. Atiyah, *Convexity and commuting Hamiltonians*, Bull. London Math. Soc. **14** (1982), no. 1, 1–15. doi:10.1112/blms/14.1.1.

[2] D. A. Cox, J. B. Little, and H. K. Schenck, *Toric Varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011.

[3] T. Delzant, *Hamiltoniens périodiques et images convexes de l'application moment*, Bull. Soc. Math. France **116** (1988), no. 3, 315–339. Numdam article.

[4] J. Dugundji, *An extension of Tietze's theorem*, Pacific J. Math. **1** (1951), no. 3, 353–367. Original article.

[5] V. Guillemin and S. Sternberg, *Convexity properties of the moment mapping*, Invent. Math. **67** (1982), no. 3, 491–513. doi:10.1007/BF01398933.

[6] J. R. Munkres, *Topology*, second edition, Prentice Hall, Upper Saddle River, NJ, 2000.

[7] Á. Pelayo, A. R. Pires, T. S. Ratiu, and S. Sabatini, *Moduli spaces of toric manifolds*, Geom. Dedicata **169** (2014), 323–341. doi:10.1007/s10711-013-9858-x. arXiv:1207.0092.

[8] Á. Pelayo and F. Santos, *Moduli spaces of Delzant polytopes and symplectic toric manifolds*, Adv. Math. **482**, Part B (2025), 110624. doi:10.1016/j.aim.2025.110624. arXiv:2303.02369v2. Numbered references to this work use the cited arXiv version.

[9] G. C. Shephard and R. J. Webster, *Metrics for sets of convex bodies*, Mathematika **12** (1965), no. 1, 73–88. doi:10.1112/S0025579300005179.

# A A quantitative disk filling

For loops, the extension can be carried out on a countable triangulation of the open disk. We give the details to express the boundary estimate directly in terms of the modulus of continuity of the loop. The construction works in every dimension $n$.

Write a loop as $f : \mathbb{R}/\mathbb{Z} \to \mathcal{D}(n)$ and define

$$\omega_f(\delta) = \sup_{d_{\mathbb{R}/\mathbb{Z}}(\theta,\phi)\le\delta} d_{\mathrm{H}}(f(\theta), f(\phi)),$$

where $d_{\mathbb{R}/\mathbb{Z}}$ is the quotient distance induced by the usual metric on $\mathbb{R}$. Compactness of the circle gives $\omega_f(\delta) \to 0$ as $\delta \downarrow 0$. Use polar coordinates $x = (1-s)e^{2\pi i\theta}$, with $s = 0$ on the boundary, and set

$$s_k = 2^{-k}, \qquad N_k = 2^{k+2} \qquad (k \ge 1).$$

On the circle $s = s_k$, use the vertices $\theta = j/N_k$. In each parameter rectangle

$$[j/N_k, (j+1)/N_k] \times [s_{k+1}, s_k],$$

join the midpoint of its outer side $s = s_{k+1}$ to both corners of its inner side $s = s_k$. The resulting three triangles agree with the doubled subdivision on the next circle. Identify $\theta = 0$ with $\theta = 1$, and fill the central disk $s \ge s_1$ by coning its subdivided boundary to the center. This is a locally finite topological triangulation of the open disk: every compact subset meets only finitely many triangles. Affine coordinates are taken in the parameter triangles and the central cone triangles.

Label each circle vertex by $f(\theta)$, and label the center by any $P_* \in \mathcal{D}(n)$. For every edge $e$, choose a Delzant label $Q_e$ whose fan refines the fans of its endpoint labels. If the minimum of the circle levels of its endpoints is $k$, require $Q_e$ to be within $2^{-k}$ of the value of $f$ at one of its endpoints. For an edge incident to the center, use any positive error and the target at its other endpoint. These choices are possible by Lemma 2.3.

For each triangle $T$ in the $k$th annulus, choose a label $Q_T$ whose fan refines those of all its proper-face labels and which is within $2^{-k}$ of the value of $f$ at one of its vertices. For central triangles choose any positive error and a target at one of their circle vertices. Again Lemma 2.3 applies. Interpolate on the barycentric subdivision. Every small triangle corresponds to a flag

$$\text{vertex} \subset \text{edge} \subset \text{triangle},$$

so Lemma 2.1 makes the resulting map $F : \operatorname{int} B^2 \to \mathcal{D}(n)$ well defined and continuous.

Suppose $x = (1-s)e^{2\pi i\theta}$ lies in the $k$th annulus. Every contributing label is within $2^{-k}$ of a value of $f$ at angular distance at most $1/N_k$ from $\theta$; vertex labels have zero error, and edges on the outer circle have the smaller error $2^{-(k+1)}$. By (2.2),

$$d_{\mathrm{H}}(F((1-s)e^{2\pi i\theta}), f(\theta)) \le 2^{-k} + \omega_f(1/N_k) \le 2s + \omega_f(s/2). \tag{A.1}$$

For the last inequality, use $s \ge 2^{-(k+1)}$ and $1/N_k = 2^{-(k+2)}$. The bound tends to zero uniformly in $\theta$ as $s \downarrow 0$. Defining $F(e^{2\pi i\theta}) = f(\theta)$ therefore extends $F$ continuously to the closed disk. Only finitely many labels, and hence finitely many normal fans, occur on every compact subset of its interior.