Quantum Speedup for Hypergraph Sparsification

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Background and Terminology

Graph: $G = (V, F, c)$, edges F with weights c

Hypergraph: $H = (V, E, w)$, hyperedges E (can connect >2 nodes)

Laplacian Quadratic Form: $x^\top L_G x = \sum_{i,j} c_{ij}(x_i - x_j)^2$

Hypergraph Energy: $Q_H(x) = \sum_e w_e \max_{i,j \in e}(x_i - x_j)^2$

Spectral Sparsification:

Given a graph or hypergraph H, an ϵ-spectral sparsifier $\widetilde{H}$ is a reweighted subgraph satisfies $|Q_H(x) - Q_{\widetilde{H}}(x)| \le \epsilon \cdot Q_H(x)$ for all $x \in \mathbb{R}^V$,

where $Q_H(x)$ denotes the Laplacian quadratic form (graph) or energy (hypergraph).

We assume query access to a hypergraph via a quantum oracle $\mathcal{O}_H$, composed of:

- $\mathcal{O}_H^{\text{size}}$ returns the size of a hyperedge
- $\mathcal{O}_H^{\text{vtx}}$ returns the list of vertices in a hyperedge
- $\mathcal{O}_H^{\text{wt}}$ returns the weight of a hyperedge

Previous Work and Main Results

Central question proposed by Apers & de Wolf (2020).:

"Is there a hypergraph sparsification algorithm that enables quantum speedups?"

Table 1. Summary of results on hypergraph sparsification

Reference	Type	Sparsifier size	Time Complexity
Soma & Yoshida (2019)	Classical	$O(n^3 \log n/\varepsilon^2)$	$\widetilde{O}(mnr + n^3/\varepsilon^2)$
Bansal et al. (2019)	Classical	$O(r^3 n \log n/\varepsilon^2)$	$\widetilde{O}(mr^2 + r^3 n/\varepsilon^2)$
Kapralov et al. (2021)	Classical	$nr(\log n/\varepsilon)^{O(1)}$	$O(mr^2) + n^{O(1)}$
Kapralov et al. (2022)	Classical	$O(n \log^3 n/\varepsilon^4)$	$\widetilde{O}(mr + \text{poly}(n))$
Jambulapati et al. (2023); Lee (2023)	Classical	$O(n \log n \log r/\varepsilon^2)$	$\widetilde{O}(mr)^\dagger$
This work	Quantum	$O(n \log n \log r/\varepsilon^2)$	$\widetilde{O}(r\sqrt{mnr} + r\sqrt{mn}/\varepsilon)$

† This $\widetilde{O}(mr)$ complexity corresponds to the algorithm proposed in Jambulapati et al. (2023).

Our algorithm achieves quantum hypergraph sparsification by combining the following components:

Quantum Leverage Score Overestimation

We extend the chaining framework of Jambulapati et al. (2023) and use quantum graph sparsification (Apers et al., 2022) on sparse underlying graphs to estimate hyperedge importance in $\widetilde{O}(r\sqrt{mnr})$ time.

Quantum Sampling via State Preparation

Using the "prepare many copies" technique (Hamoudi et al., 2022), we sample hyperedges proportional to their importance without computing normalization, achieving $\widetilde{O}(r\sqrt{mn}/\epsilon)$ runtime.

Reweighting with Quantum Sum Estimation

We apply quantum sum estimation (Li et al., 2019) and a chaining argument (Lee and Sun, 2023) to reweight sampled edges and ensure spectral accuracy.

Algorithm: Quantum Hyperedge Leverage Score Overestimates

To sparsify a hypergraph, we define the leverage score ℓ_e of a hyperedge e as $\ell_e := w_e \cdot \max_{f \in \binom{e}{2}} R_f$, where R_f is the effective resistance of edge f in an underlying graph of the hypergraph. This underlying graph replaces each hyperedge by a clique of edges with redistributed weights.

Algorithm 1 Quantum Hyperedge Leverage Score Overestimates $\mathsf{QHLSO}(\mathcal{O}_H, T, \alpha_1, \alpha_2)$

Require: Quantum Oracle $\mathcal{O}_H$ to a hypergraph $H = (V, E, w)$ with $|V| = n, |E| = m$, rank r; the number of episodes $T \in \mathbb{N}$; positive real numbers $\alpha_1, \alpha_2 \in \mathbb{R}$.

Ensure: An instance $\mathcal{Z}$ of QOverestimate which stores the vector z being an $O(n)$-overestimate for H.

Let $U_{G(1)} = \mathsf{WeightInitialize}(\mathcal{O}_H)$.
for $t = 1$ to T **do**
  $\widetilde{G}^{(t)} = (V, \widetilde{F}^{(t)}, \widetilde{c}^{(t)}) \leftarrow \mathsf{GraphSparsify}(U_{G(t)}, \alpha_1)$.
  $\mathcal{G}^{(t)} \leftarrow \mathsf{UGraphStore}(\widetilde{G}^{(t)})$.
  $\mathcal{R}^{(t)} \leftarrow \mathsf{EffectiveResistance}(\widetilde{G}^{(t)}, \alpha_2)$.
  $U_{G(t+1)} = \mathsf{WeightCompute}(\mathcal{O}_H, \mathcal{R}^{(t)}, \mathcal{G}^{(t)})$.
end for
$C_1 \leftarrow 2(1 + \frac{\alpha_1 + \alpha_2}{1 - \alpha_1}) \cdot \exp(\log r/T)$.
$\mathcal{Z} \leftarrow \mathsf{QOverestimate}(\{\mathcal{G}^{(t)} : t \in [T]\}, \{\mathcal{R}^{(t)} : t \in [T]\}, \mathcal{O}_H, C_1, T)$.

Theorem 1

There exists a quantum algorithm that, given query access to a hypergraph with n vertices, m hyperedges, and rank r, constructs (with high probability) a data structure enabling queries to an $O(n)$-bounded leverage score overestimate vector z, where each z_e can be computed in $\widetilde{O}(r)$ time. The total preprocessing time is $\widetilde{O}(r\sqrt{mnr})$.

Algorithm: Quantum Hypergraph Sparsification

Algorithm 2 Quantum Hypergraph Sparsification $\mathsf{QHypergraphSparse}(\mathcal{O}_H, \epsilon)$

Require: Quantum Oracle $\mathcal{O}_H$ to a hypergraph $H = (V, E, w)$ with $|V| = n, |E| = m$, rank r; accuracy $\epsilon > 0$.

Ensure: An ϵ-spectral sparsifier of H, denote by $\widetilde{H} = (V, \widetilde{E}, \widetilde{w})$, $|\widetilde{E}| = O(n \log n \log r/\epsilon)$.

$\widetilde{E} = \varnothing, \widetilde{w} = 0, M \leftarrow \Theta\left(n \log n \log r/\epsilon^2\right)$.
$\mathcal{Z} \leftarrow \mathsf{QHLSO}(\mathcal{O}_H, \log(r-1), 0.1, 0.1)$.
$\sigma \leftarrow \mathsf{MultiSample}(\mathcal{Z}.\mathsf{Query}, M)$.
$s \leftarrow \mathsf{SumEstimate}(\mathcal{Z}.\mathsf{Query}, 0.1)$.
for $i = 1$ to M **do**
  $w_{\sigma_i} \leftarrow$ measurement outcome of the second register of $\mathcal{O}_H^{\text{wt}}|\sigma_i\rangle|0\rangle$.
  $z_{\sigma_i} \leftarrow$ measurement outcome of the second register of $\mathcal{Z}.\mathsf{Query}|\sigma_i\rangle|0\rangle$.
  $\widetilde{E} \leftarrow \widetilde{E} \cup \{\sigma_i\}, \widetilde{w}_{\sigma_i} \leftarrow \widetilde{w}_{\sigma_i} + w_{\sigma_i} \cdot s/(Mz_{\sigma_i})$.
end for

Theorem 2

There exists a quantum algorithm that, given query access to a hypergraph with n vertices, m hyperedges, and rank r, outputs an ϵ-spectral sparsifier with $\widetilde{O}(n/\epsilon^2)$ hyperedges in time $\widetilde{O}(r\sqrt{mnr} + r\sqrt{mn}/\epsilon)$.

Applications

Quantum hypergraph sparsification enables faster approximation algorithms for classical cut problems in hypergraphs. In particular, we obtain sublinear-time quantum algorithms for the following tasks:

Corollary 1 – Hypergraph Cut Sparsification

An ϵ-cut sparsifier with $\widetilde{O}(n/\epsilon^2)$ hyperedges can be constructed in $\widetilde{O}(r\sqrt{mnr} + r\sqrt{mn}/\epsilon)$ time. This enables efficient downstream cut computations.

Corollary 2 – Hypergraph Mincut

A $(1+\epsilon)$-approximate global mincut can be computed in $\widetilde{O}(r\sqrt{mnr} + r\sqrt{mn}/\epsilon + rn^2/\epsilon^2)$ time.

Corollary 3 – s-t Mincut

For nodes $s, t \in V$, an approximate s-t mincut can be found in $\widetilde{O}(r\sqrt{mnr} + r\sqrt{mn}/\epsilon + rn^{3/2}/\epsilon^3)$ time.

Open Questions

- Can we design fast quantum algorithms for more hypergraph problems like k-cut, spectral diffusion, or min-max k-partition?
- Can we prove a tight lower bound of $\Omega(r\sqrt{mn}/\epsilon)$ or further reduce the dependence on ϵ in quantum hypergraph sparsification?
- Can quantum speedups extend to advanced sparsification settings—online, directed, submodular, or even GLM sparsification?

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