Quantum Speedup for Sampling Random Spanning Trees

Chenghua Liu[1] Minbo Gao[1] Zhengfeng Ji[2] Simon Apers[3]

[1] Institute of Software, Chinese Academy of Sciences, Beijing, China

[2] Department of Computer Science and Technology, Tsinghua University, Beijing, China

[3] Université Paris Cité, CNRS, IRIF, Paris, France

July 10, 2025 @ ICALP 2025, Aarhus, Denmark

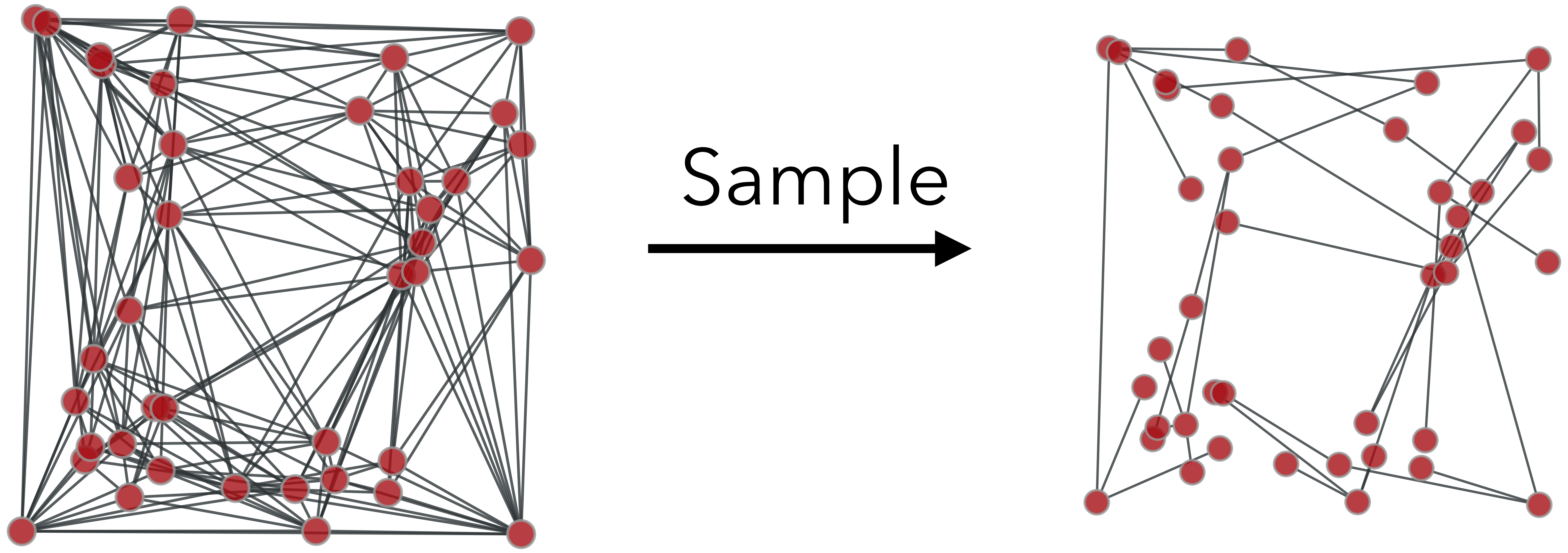
Random Spanning Trees-1

$G = (V, E, w)$

Let $\mathscr{T}_G$ denote the set of all spanning trees of G

Define the distribution W_G on $\mathscr{T}_G$ by $P_{X \sim W_G}[X = T] \propto \prod_{e \in T} w_e$

Goal: sample a spanning tree $T \sim W_G$



Random Spanning Trees

Many Applications: algorithm design, machine learning, statistics …

Sampling RSTs has been widely studied classically, with three main algorithmic approaches:

1. **Determinant-Based Methods:**

Guenoche 83 & Kulkarni 90: $O(mn^3)$;

Colbourn, Myrvold, and Neufeld 96: $O(n^\omega)$.

2. **Effective Resistance-Based Methods:**

Harvey and Xu 16: $O(n^\omega)$;

Durfee, Kyng, Peebles, Rao and Sachdeva 17: $\widetilde{O}(n^{4/3}m^{1/2} + n^2)$;

Durfee, Peebles, Peng and Rao 17: $\widetilde{O}(n^2/\varepsilon^2)$.

3. **Random Walk-Based Methods:**

Broder 89 & Aldous 90: $O(mn)$ for unweighted;

Wilson 96; Kelner, Madry 09; Madry, Straszak,Tarnawski 14; Schild 18: $m^{1+o(1)}$;

The current state-of-the-art: based on down-up random walks, $O(m \log^2 m)$.

Our Results

$G = (V, E, w),\ |V| = n, |E| = m, w \in \mathbb{R}_+^E$

$$P_{X \sim W_G}[X = T] \propto \prod_{e \in T} w_e$$

Theorem.

There exists a quantum algorithm that, given query access to the adjacency list of a connected graph G and accuracy parameter ε, with high probability, outputs a spanning tree of G drawn from a distribution which is ε-close to W_G in total variation distance. The algorithm runs in $\widetilde{O}(\sqrt{mn}\log(1/\varepsilon))$ time.

Lower bound:

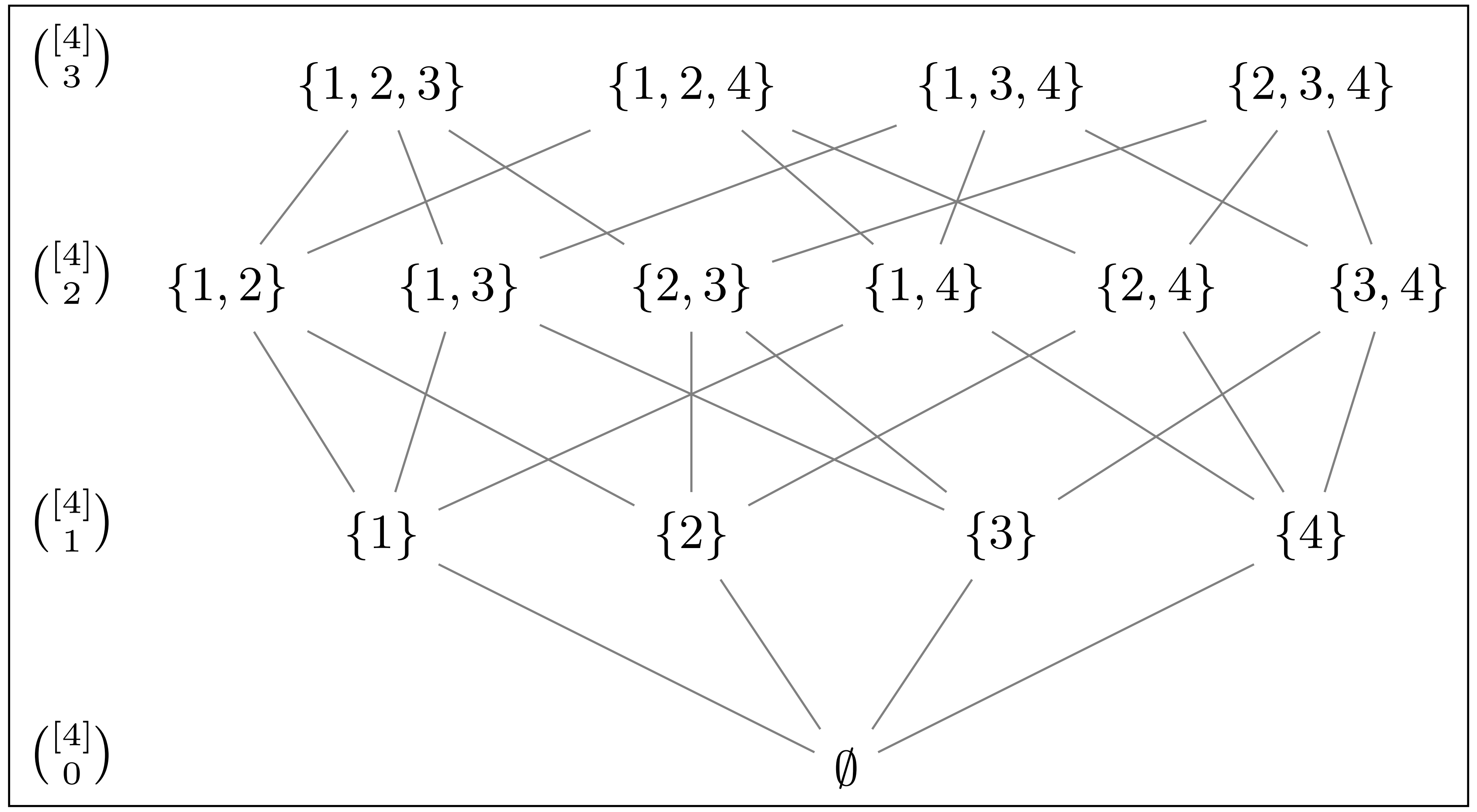
Let $\varepsilon < 1/2$ be a constant. For any graph G, consider the problem of sampling a random spanning tree from a distribution ε-close to W_G, given adjacency-list access to G. The quantum query complexity of this problem is $\Omega(\sqrt{mn})$.

Background: down-up walk

Consider a distribution μ over size-k subsets of $[m]$

We define the down operator$D_{k\to\ell}$ and the up operator $U_{\ell\to k}$, which map between sets of different sizes: from size-k to size-ℓ subsets, and vice versa.

$$U_{\ell\to k}(T,S) = \begin{cases} 0 & \text{if } T \nsubseteq S, \\ \dfrac{\mu(S)}{\sum_{S':T\subseteq S'} \mu(S')} & \text{otherwise} \end{cases} \qquad D_{k\to\ell}(S,T) = \begin{cases} 0 & \text{if } T \nsubseteq S, \\ \dfrac{1}{\binom{k}{\ell}} & \text{otherwise} \end{cases}$$



The **down operator** randomly selects a smaller subset (moving downward). And the **up operator** moves upward by choosing a size-k superset with probability proportional to $\mu(S)$.

Down-up walk for sampling RSTs

Starting from $S_0 \in \text{supp}(\mu)$, one step of the down-up walk M_μ^t , $t \geq k+1$:

1. Sample $T \in \binom{[m]\backslash S_0}{t-k}$ uniformly at random

2. Let $S_1 \sim \mu_{S_0 \cup T}$, and update $S_0 \leftarrow S_1$

$\mu_{S_0 \cup T}$ is μ restricted to $S_0 \cup T$

Lemma. Proposition 25 in [ADVY22]

The complement of S_1 is distributed according to $\bar{\mu}_0 D_{(m-k)\to(m-t)} U_{(m-t)\to(m-k)}$ if we start with $S_0 \sim \mu_0$, where $\bar{\mu}(S) := \mu([m] \setminus S)$. Moreover, for any distribution μ that is strongly Rayleigh, the chain is irreducible, aperiodic and has stationary distribution μ.

W_G is strongly Rayleigh, and 1-step down-up walk becomes:

Remove an edge uniformly randomly

Add a new edge sampled proportionally to the edge weights between components

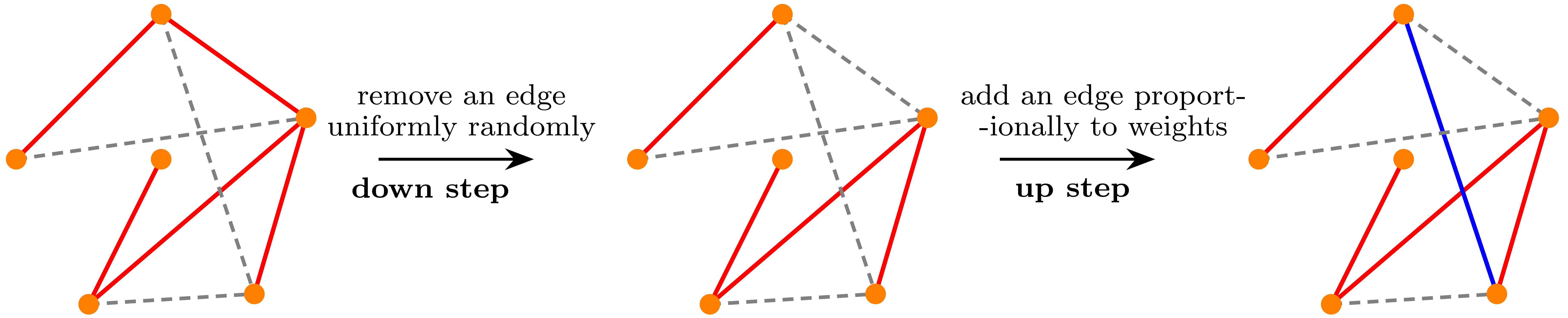
Down-up walk for sampling RSTs

[Nima, Kuikui, Shayan, Cynthia, Thuy-Duong STOC21]

1-step down-up walk for sampling RSTs:

1. remove an edge to split the tree
2. add a new edge sampled proportionally to the edge weights between components

The chain has the mixing time $\widetilde{O}(n)$.



Their final algorithm uses an "up-down" walk: first add an edge, then remove one. Although the mixing time is $\widetilde{O}(m)$, each sample step runs in amortized $\widetilde{O}(1)$ time via link-cut trees.

Barrier and Idea for Quantum Speedups

Revisit the Down-Up Walk:
We can sample an edge between the two resulting components in $\widetilde{O}(\sqrt{m})$ time using Grover Search. But overall complexity remains $\widetilde{O}(\sqrt{mn}) \in \widetilde{\Omega}(m)$, offering no speedup.

Inspired by domain sparsification techniques [AD20, ADVY22, ALV22]

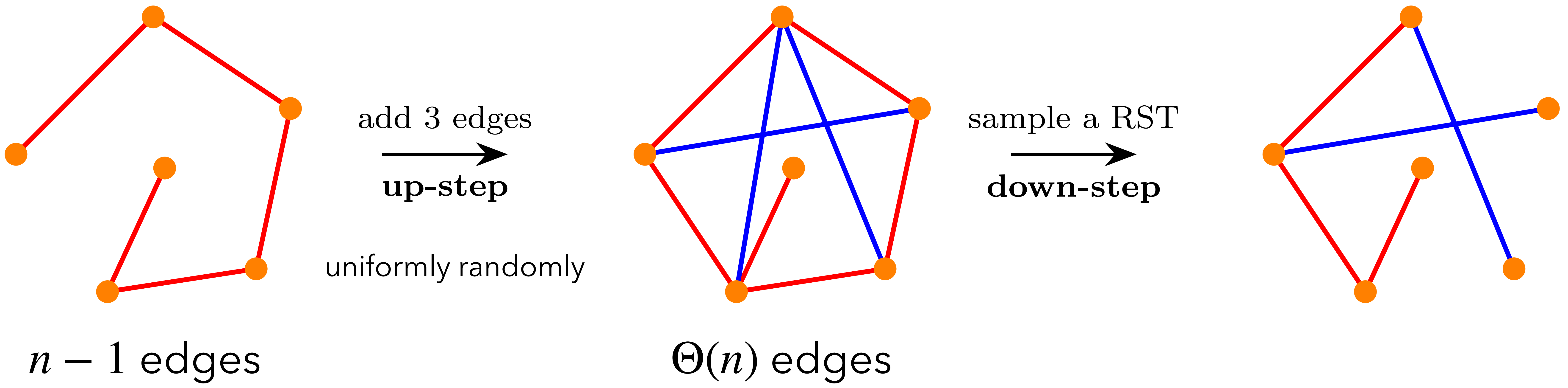
Key Ideas:

- *Large-Step Walks*: Modify $\Theta(n)$ edges in each step (vs. 1 in classical), reducing mixing time to $\widetilde{O}(1)$.
- *Isotropy transformation*: Reduce sampling domain size from m to $O(n)$ using isotropic transformation (enables large-step walks to work well).

Quantum Sampling RSTs

Framework.

Large-Step Walks: Modify $\Theta(n)$ edges in each step, reducing mixing time to $\widetilde{O}(1)$.



Requirement.

Perform an isotropic transformation–that is, adjust the graph so that the marginal probabilities of all edges are approximately equal.

The marginal probability of an edge e is given by: $\Pr[e \in T, T \sim W_G] = w_e \cdot R(e)$

$R(e) := (\delta_i - \delta_j)^\top L_G^+(\delta_i - \delta_j),\ e = \{i, j\}$

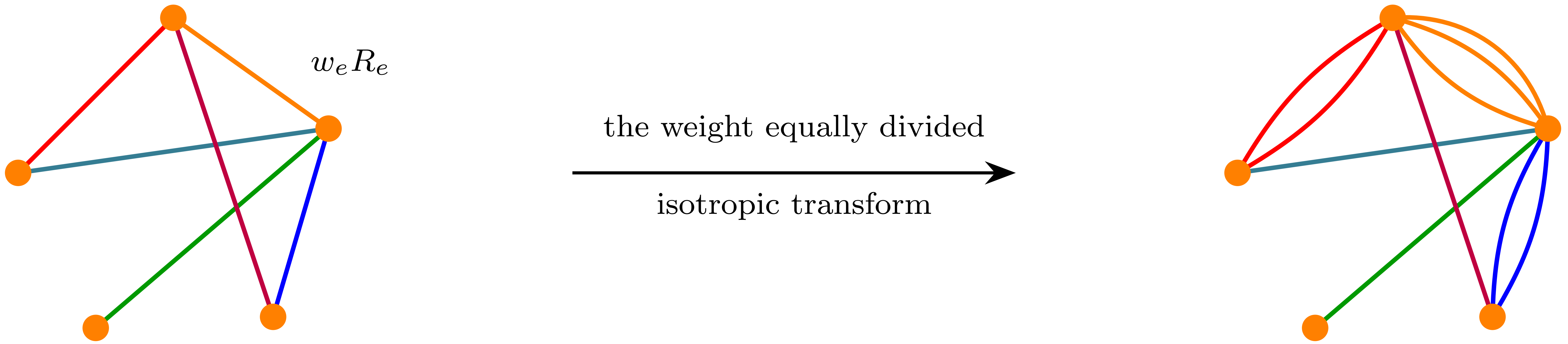
Isotropic Transformation

Defintion.

Given a graph $G = (V, E, w)$ and a vector $\widetilde{R} \in \mathbb{R}^E$ approximating effective resistances R, the isotropic-transformed multigraph $G' = (V, E', w')$ is constructed as follows:

Each edge $e \in E$ is replaced by $q_e = \lceil m \cdot w_e \widetilde{R}_e / (2n) \rceil$ parallel edges.

Each copy has weight w_e / q_e.



Proposition.

The transformed graph satisfies $|E'| \leq 2m$, and the marginals are nearly uniform:

$\Pr[e' \in S, S \sim \mathscr{W}_{G'}] \leq 2n/m = o(1)$.

Then the mixing time is $\widetilde{O}(1)$, by the analysis in [ALV22].

Implicit Isotropic Transformation

Quantum Graph Sparsification [AdW22] Time $\widetilde{O}(\sqrt{mn})$

Rather than explicitly computing this isotropic transformation, we utilize a quantum data structure $\mathscr{R}$ which provides quantum query access to effective resistances, to ``implicitly'' construct and maintain the isotropic-transformed multigraph.

Quantum Isotropic Sampling with $\mathscr{R}$ (up step) Time $\widetilde{O}(\sqrt{mn})$

Sample $\Theta(n)$ edges from the isotropic-transformed multigraph uniformly at random (a sampling-without-replacement variant of multiple-state preparation **[Ham22]**).

Quantum Minimum Spanning Tree [DHHM06] Time $\widetilde{O}(\sqrt{mn})$

Find a spanning tree with maximum product of edge weights as a “good” starting point for the down-up walk.

Quantum Lower Bound

Lower bound:

Let $\varepsilon < 1/2$ be a constant. For any graph G, consider the problem of sampling a random spanning tree from a distribution ε-close to W_G. The quantum query complexity of this problem is $\Omega(\sqrt{mn})$.

Follows via reduction from finding n marked elements among m, which has quantum query complexity $\Theta(\sqrt{mn})$. The reduction encodes the search into edge weights so that a uniform spanning tree reveals the marked elements.

Similar to the $\Omega(\sqrt{mn})$ lower bound for MST in [DHHM06].

Open Questions

- Faster algorithm for unweighted graphs?
- The down-up walk is a powerful tool in classical algorithms (e.g., colorings, matchings). Can our quantum approach yield speedups for them?
- Determinantal Point Processes (DPPs)?

Thanks for your attention !